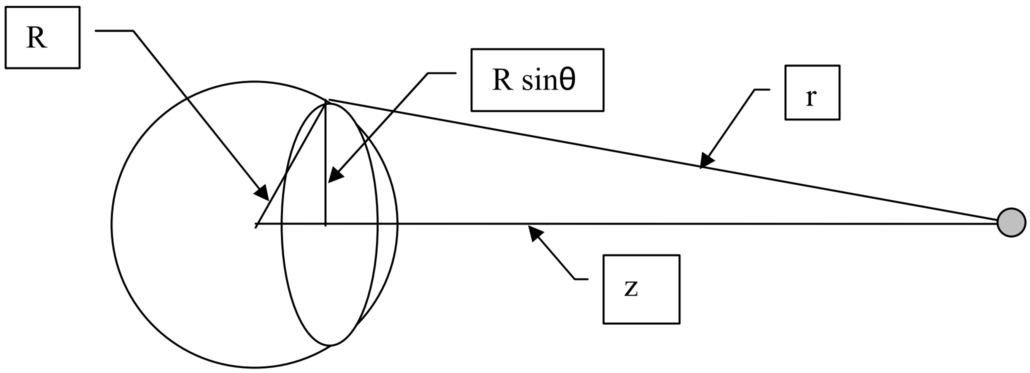
SHELL THEOREM PROOF

Consider the thin spherical shell of radius R:



Let ρ be the density (mass per unit area) of the shell, M be the mass of the shell, and m be the mass of a small test object a distance z from the center of the shell.

$$dM = 2\pi\rho \; R\sin\theta \; R\,d\theta$$

$$r^2 = (z - R\cos\theta)^2 + R^2\sin^2\theta$$

$$dU = -G\frac{m\,dM}{r}$$

$$U = -2\pi R^2 \rho m G \int_0^{\pi} \frac{\sin\theta d\theta}{\sqrt{z^2 - 2z\;R\cos\theta + R^2}}$$

Let x = cos θ:

$$U = -2\pi R^2 \rho m G \int_{-1}^{1} \frac{dx}{\sqrt{z^2 + R^2 - 2zRx}}$$

$$U = -\frac{1}{2} GmM \int_{-1}^{1} \frac{dx}{\sqrt{z^2 + R^2 - 2zRx}}$$

$$U = -G\frac{mM}{2} \int_{-1}^{1} \frac{dx}{\sqrt{a + bx}}$$

where: $a = z^2 + R^2$ and $b = -2\,zR$.

From the tables:

$$U = -G\frac{mM}{2}\left[\frac{2\sqrt{a+bx}}{b}\right]_{-1}^{1}$$

but at the limits:

$$a + bx = (z^2 - 2zRx + R^2) \;\; = (z - xR)^2$$

where x is either+1 or -1.

Thus outside the sphere (z > R):

$$U = G\frac{mM}{2zR}\big((z - R) - (z + R)\big)$$

$$U = -G\frac{mM}{z}$$

$$\vec{F} = -\vec{\nabla}U = -G\frac{mM}{z^2}\hat{k}$$

Inside the sphere (z < R):

$$U = G\frac{mM}{2zR}\big((R \; - z) - (R \; + z \;)\big)$$

$$U = -G\frac{mM}{R}$$

$$\vec{F} = 0$$

Recall that R is the (constant) radius of the sphere and z is the distance of the object from the center of the sphere.