

1 Evaluating Line Integrals

Note that while line integrals are defined for any continuous function, the primary example we have studied involved work integrals. This is when the function we are integrating is defined as the dot product $\mathbf{F} \bullet \mathbf{T}$ of two vectors:

\mathbf{F} , a force vector

\mathbf{T} , the unit vector tangent to the curve

Method 1: Parametrize the curve

- If C is parametrized as $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \|\mathbf{r}'(t)\| dt$$

- For work integrals, if C is parametrized as above and $\mathbf{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$

$$\begin{aligned} \int_C \mathbf{F} \bullet \mathbf{T} ds &= \int_C \mathbf{F} \bullet d\mathbf{r} \\ &= \int_C f(x, y, z) dx + g(x, y, z) dy + h(x, y, z) dz \\ &= \int_a^b [(f)(x'(t)) + (g)(y'(t)) + (h)(z'(t))] dt \end{aligned}$$

Note: in the last integral above, the functions f , g , and h are expressed in terms of t , so that, for example, $f = f(x(t), y(t), z(t))$.

Method 2: For work integrals involving conservative fields only!!

- If \mathbf{F} is conservative, there exists a potential function ϕ such that $\mathbf{F} = \nabla\phi$.
- If C is a curve from P_0 to P_1 then $\int_C \mathbf{F} \bullet \mathbf{T} ds = \int_C \mathbf{F} \bullet d\mathbf{r} = \phi(P_1) - \phi(P_0)$.

Method 3: For work integrals over the boundary of a piecewise smooth oriented surface

- Use Stokes' theorem to convert to a surface integral:

$$\int_C \mathbf{F} \bullet \mathbf{T} ds = \iint_{\sigma} (\text{curl}\mathbf{F}) \bullet \mathbf{n} dS$$

where

σ is a piecewise smooth oriented surface

C is its boundary, with positive orientation

2 Evaluating Surface Integrals

Note that while surface integrals are defined for any continuous function, we have primarily worked with flux integrals, in which the function we are integrating is defined as the dot product $\mathbf{F} \cdot \mathbf{n}$ in which \mathbf{F} is a vector field and \mathbf{n} the unit vector normal to the surface.

Method 1: Parametrize the surface

Surfaces parametrized in terms of u and v If σ is the surface parametrized by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ over the region R in the uv -plane

- General surface integral:

$$\iint_{\sigma} f(x, y, z) dS = \iint_R f(x(u, v), y(u, v), z(u, v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA$$

- Flux integral:

$$\iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_R \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dA$$

Note that if we wish to choose our normal vector to point in the opposite direction, we can simply replace $\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right)$ with $-\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right)$.

Surfaces defined by $z = f(x, y)$ over a region R in the xy -plane Note that this function defines a parametrization

$\mathbf{r}(u, v) = \langle u, v, f(u, v) \rangle$, where $u = x$ and $v = y$

In this case $\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right)$ (expressed in terms of x and y) is $\langle -f_x, -f_y, 1 \rangle$

- General surface integral:

$$\iint_{\sigma} g(x, y, z) dS = \iint_R g(x, y, f(x, y)) \sqrt{(f_x)^2 + (f_y)^2 + 1} dA$$

- Flux integrals:

$$\Phi = \iint_{\sigma} \mathbf{F} \bullet \mathbf{n} \, dS = \iint_R \mathbf{F} \bullet \langle -f_x, -f_y, 1 \rangle \, dA$$

Note that \mathbf{n} would be $\langle -f_x, -f_y, 1 \rangle$ normalized, so that our surface is oriented so that the normal vector has a positive z-component (it points up, though not necessarily straight up.) If we wish to orient the surface so that the normal points down instead, we would simply replace $\langle -f_x, -f_y, 1 \rangle$ with $\langle f_x, f_y, -1 \rangle$.

- Note also that we could define the surface above to be the level surface $G(x,y,z) = z - f(x,y) = 0$. In this case we could also calculate \mathbf{n} to be $\frac{\nabla G}{\|\nabla G\|}$, and the flux integral becomes

$$\Phi = \iint_{\sigma} \mathbf{F} \bullet \mathbf{n} \, dS = \iint_R \mathbf{F} \bullet \nabla G \, dA$$

- Similar techniques work if the surface is defined by $x = g(y,z)$ or $y = g(z,x)$ (Note in this last example that we write "z" first, so that in parametrizing, $u = z$ (not x) and $v = x$ (not z). This way the parametrization describes a surface with the normal pointing in the positive y-direction; the other way the parametrization describes a surface with the normal pointing in the negative y-direction.) See the table on page 1142 of your text for the normal vectors we get in each of these cases.

Method 2: For flux integrals over the boundary of a finite solid

Use the divergence theorem to pass to a triple integral over the enclosed solid. If σ is the boundary of a finite solid, G , with \mathbf{n} the unit normal pointing *out* (away from the solid):

$$\iint_{\sigma} \mathbf{F} \bullet \mathbf{n} \, dS = \iiint_G \operatorname{div} \mathbf{F} \, dV$$

Method 3: For integrals of the form $\iint_{\sigma} (\operatorname{curl} \mathbf{F}) \bullet \mathbf{n} \, dS$

Use Stokes' theorem to pass to a work integral over the boundary. If C is the boundary, oriented positively with respect to σ :

$$\iint_{\sigma} (\operatorname{curl} \mathbf{F}) \bullet \mathbf{n} \, dS = \oint_C \mathbf{F} \bullet \mathbf{T} \, ds$$

Note: C must be simple, closed, and piecewise smooth.