Solutions: Math 3c, Exam 1 Fall '06 Karla Westphal

1. Describe geometrically the shape of each graph in 3-space. One-word answers (plane, sphere, etc.) will suffice.

   (a) 3x-4y+2z= 6
   Plane.

   (b) $\mathbf{r}(t) = \langle 1,1,1 \rangle + t\,\langle 0,-4,2 \rangle$
   Line.

   (c) x=2
   Plane. (This plane is parallel to the yz-plane, but 2 units in front of it.)

2. Sketch the graph a parametric curve in $\mathbb{R}^2$such that all of the following hold. Label the points that correspond to t = -5 and $t = -3$ and remember to include the orientation.

   $x'(t) < 0 \quad if\ t < -3$
   $x'(t) = 0 \quad if\ t = -3$
   $x'(t) > 0 \quad if\ t > -3$
   $y'(t) < 0 \quad if\ t < -5$
   $y'(t) = 0 \quad if\ t = -5$
   $y'(t) > 0 \quad if\ t > -5$

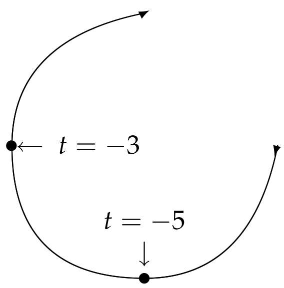
   At $t = -5$: $y'(t) = 0; x'(t) \neq 0$ so we have a horizontal tangent.

   At $t = -3$: $x'(t) = 0; y'(t) \neq 0$ so we have a vertical tangent.

   The picture below shows what the given information tells us about how the graph is traced on the three intervals determined by -5 and -3.

	-5		-3	
$x'(t) < 0 \Rightarrow$ *left*		$x'(t) < 0 \Rightarrow$ *left*		$x'(t) > 0 \Rightarrow$ *right*
$y'(t) < 0 \Rightarrow$ *down*		$y'(t) > 0 \Rightarrow$ *up*		$y'(t) > 0 \Rightarrow$ *up*

   There are infinitely many possible answers. Below is one of them.



   Note: The orientation is shown with arrows, but the arrows may be hard to read. The graph is traced clockwise.

3. Consider the parametric curve: $x(t) = t^2 - 2$
$y(t) = \sin t$

(a) Find a formula for $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{2t}$

(b) Find a formula for $\frac{d^2y}{dx^2}$.

Letting y' denote $\frac{dy}{dx} = \frac{\cos t}{2t}$ (from above):

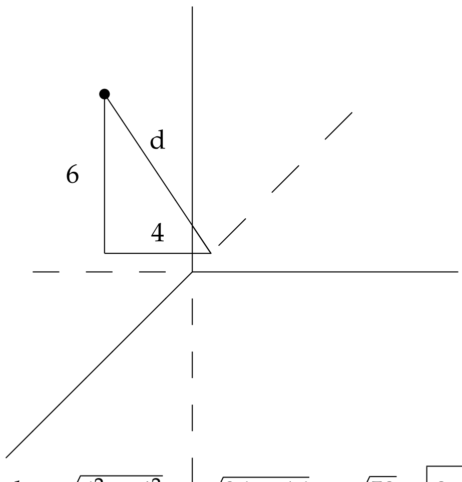
$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{2t(-\sin t) - \cos t(2)}{(2t)^2}}{2t} = \frac{-2t \sin t - 2\cos t}{(4t^2)(2t)} = \frac{-2t\sin t - 2\cos t}{8t^3} = \frac{2(-t\sin t - \cos t)}{8t^3} = \frac{-t\sin t - \cos t}{4t^3}$

4. Consider the point P(-1, -4, 6).

(a) What is the distance from this point to the xz-plane?

distance to xz-plane = $|-4| = 4$

(b) What is the distance from this point to the x-axis?



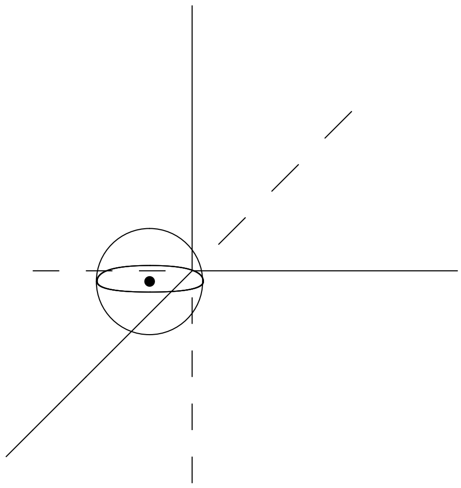
$d = \sqrt{6^2 + 4^2} = \sqrt{36 + 16} = \sqrt{52} = \boxed{2\sqrt{13}}$

5. Sketch in 3-space:

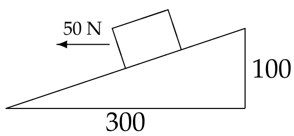
$$(x-1)^2 + (y+3)^2 + z^2 = 16$$

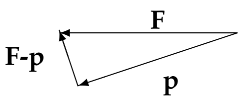
You may supplement your picture with words if necessary to clarify what you are trying to draw.

The graph is a sphere of radius 4, centered at (1, -3, 0).



6. As shown below, a force of 50 Newtons acts on an object lying on an inclined plane. Find the vector components of force acting parallel and perpendicular to the plane.





$\mathbf{F} = \left\langle -50, 0 \right\rangle$

$\mathbf{u}$ = unit vector parallel to plane = $\dfrac{\left\langle -3, \quad -1 \right\rangle}{\sqrt{3^2+1^2}} = \left\langle -\frac{3}{\sqrt{10}}, \quad \frac{-1}{\sqrt{10}} \right\rangle$

$\mathbf{p}$ = vector component parallel to plane = projection onto $\mathbf{u}$ = $(\mathbf{f} \bullet \mathbf{u})\, \mathbf{u} =$

$$\underbrace{\left(\left\langle -50, \quad 0 \right\rangle \bullet \left\langle \frac{-3}{\sqrt{10}}, \quad \frac{-1}{\sqrt{10}} \right\rangle \right)}_{scalar} \left\langle \frac{-3}{\sqrt{10}}, \quad \frac{-1}{\sqrt{10}} \right\rangle = \left(\frac{150}{\sqrt{10}} \right) \left\langle \frac{-3}{\sqrt{10}}, \quad \frac{-1}{\sqrt{10}} \right\rangle = \left\langle \frac{-450}{10}, \quad \frac{-150}{10} \right\rangle =$$

$\boxed{\langle -45, \quad -15 \rangle}$

F - p = vector component perpendicular to the plane = $\langle -50, \quad 0 \rangle - \langle -45, \quad -15 \rangle =$

$\boxed{\langle -5, \quad 15 \rangle}$

7. Let $\mathbf{u} = \langle u_1, \quad u_2 \rangle$, $\mathbf{v} = \langle v_1, \quad v_2 \rangle$, and $\mathbf{w} = \langle w_1, \quad w_2 \rangle$ be three vectors in $\mathbb{R}^2$. Prove:

$$\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$$

$$\begin{aligned}
\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) &= \langle u_1, \quad u_2 \rangle \bullet (\langle v_1, \quad v_2 \rangle + \langle w_1, \quad w_2 \rangle) \\
&\underbrace{=}_{\text{def. of vector addition}} \langle u_1, \quad u_2 \rangle \bullet \langle v_1 + w_1, \quad v_2 + w_2 \rangle \\
&\underbrace{=}_{\text{def. of dot product}} u_1(v_1 + w_1) + u_2(v_2 + w_2) \\
&\underbrace{=}_{\text{distributive property of real numbers}} (u_1v_1 + u_1w_1) + (u_2v_2 + v_2w_2) \\
&\underbrace{=}_{\text{commutative and associative laws of real numbers—the commutative law allows us to change the order; the associative law allows us to regroup the terms}} (u_1v_1 + u_1v_2) + (u_1w_1 + u_2w_2) \\
&\underbrace{=}_{\text{def. of dot product}} \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}
\end{aligned}$$

8. Find the volume of the parallelpiped with adjacent sides: $\mathbf{u} = \langle 2, \quad -3, \quad 0 \rangle$
$\mathbf{v} = \langle 1, \quad 0, \quad 4 \rangle$
$\mathbf{w} = \langle 2, \quad -1, \quad 2 \rangle$

Volume = $|det(\mathbf{A})|$, where $\mathbf{A}$ is the matrix with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as its rows.

$$\begin{vmatrix} 2 & -3 & 0 \\ 1 & 0 & 4 \\ 2 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 4 \\ -1 & 2 \end{vmatrix} - -3\begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$$

$= 2(4) + 3(-6) + 0 = 8 - 18 = -10$

Volume = $|-10|$ = $\boxed{10}$

9. Find the vector equation for the line segment from P(2, -3, 5) to Q(-3, -4, 8).

starting point: (2,-3,5)

dirction vector: $\vec{PQ} = \langle -3-2, \quad -4--3, \quad 8-5 \rangle = \langle -5, \quad -1, \quad 3 \rangle$

Line segment: $\mathbf{r}(t) = \langle 2, \quad -3, \quad 5 \rangle + t\langle -5, \quad -1, \quad 3 \rangle$; $0 \leq t \leq 1$

Don't forget to restrict t so that we only get the line segment!!

10. Find the equation of the plane through the points P(4,1,1), Q(-1, 0, 2), and R(3, -3, 2)

Point: I will work with P(4,1,1). Any of the 3 points is an acceptable choice.

Normal: To find the normal, we will take the cross product of two vectors that are parallel to the plane. Two such vectors are :

$\vec{PQ} = \langle -1-4, \quad 0-1, \quad 2-1 \rangle = \langle -5, \quad -1, \quad 1 \rangle$

$\vec{PR} = \langle 3-4, \quad -3-1, \quad 2-1 \rangle = \langle -1, \quad -4, \quad 1 \rangle$

$\vec{PQ} \times \vec{PR} =$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -1 & 1 \\ -1 & -4 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -5 & 1 \\ -1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -5 & -1 \\ -1 & -4 \end{vmatrix}\mathbf{k}$$

$= 3\mathbf{i}4\mathbf{j} + 19\mathbf{k} = \langle 3, 4, 19 \rangle$

Plane: $\langle 3, \quad 4, \quad 19 \rangle \bullet \langle x-4, \quad y-1, \quad z-1 \rangle = 0$

$\boxed{3(x-4) + 4(y-1) + 19(z-1) = 0}$