Solutions: Math 3c, Exam 1

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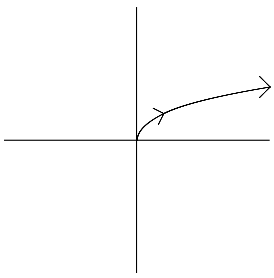
Spring 2007

Directions: Show all work and fully justify all answers. Simplify to the extent we have in class. If you have questions about whether an answer needs further justification or simplification, ASK!

1. Sketch the curve below by eliminating the parameter. Be sure to include the orientation, if there is one.

$$x = t^2$$
$$y = \sqrt{t}$$

$$y = \sqrt{t} \quad \Rightarrow \quad t = y^2; y \geq 0$$
$$x = t^2 \quad \Rightarrow \quad x = \left(y^2\right)^2 = y^4$$



2. Consider the parametric curve below:

$$x = \tan t$$
$$y = t^2 + t$$

(a) Find a formula for $\frac{dy}{dx}$.

$\frac{dx}{dt} = \sec^2 t$

$\frac{dy}{dt} = 2t + 1$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{\sec^2 t}$

(b) Find a formula for $\frac{d^2y}{dx^2}$.

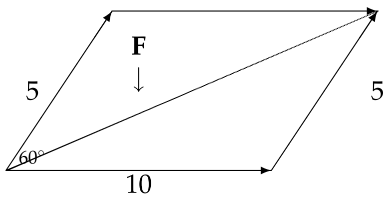
$$y' = \frac{dy}{dx} \underbrace{=}_{\text{(part a)}} \frac{2t+1}{\sec^2 t}$$

$$\frac{dy'}{dt} = \frac{2\sec^2 t - (2t+1)2\sec^2 t \tan t}{\sec^4 t} = \frac{2-(2t+1)2\tan t}{\sec^2 t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{2-(4t+2)\tan t}{\sec^2 t}}{\sec^2 t} = \frac{2-(4t+2)\tan t}{\sec^4 t}$$

3. A force of magnitude 10 and a second force of manitude 5 act on the same object. The angle between the two forces is $60°$.

   Find the magnitude of the resultant force.



In the picture above, **F** represents the resultant force.

$$||\mathbf{F}||^2 = 10^2 + 5^2 + 2(10)(5)\cos\left(\frac{\pi}{3}\right)$$

$$\begin{aligned} &= 100 + 25 + 100\left(\frac{1}{2}\right) \\ &= 175 \\ ||\mathbf{F}|| &= \sqrt{175} \\ &= 5\sqrt{7} \end{aligned}$$

4. Give the vector equation of the line through P(2, -4, 3) parallel to the line whose parametric equations are:

$$x = 4 - t$$
$$y = 3t$$
$$z = 5 + 2t$$

Starting point: $(2, -4, 3)$

Direction vector: $\langle -1, \ \ 3, \ \ 2 \rangle$ (The direction vector for our line will be the same (or a multiple of) that of the given line. To find a direction vector for the given line, just take the coefficients in front of t.)

Line: $\mathbf{L}(t) = \langle 2, \ \ -4, \ \ 3 \rangle + t \langle -1, \ \ 3, \ \ 2 \rangle$

5. (a) If **u, v** and **w** are three non-zero vectors in 3-space, state the geometric significance of the following equation:

$$\begin{vmatrix} \leftarrow & \mathbf{u} & \rightarrow \\ \leftarrow & \mathbf{v} & \rightarrow \\ \leftarrow & \mathbf{w} & \rightarrow \end{vmatrix} = 0$$

The volume of the parallelpiped whose sides are determined by the three vectors is 0. In other words, the three vectors determine a degenerate parllelpiped. Equivalently, the three vectors are coplanar (all lie in the same plane.)

(b) If $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$ is a parametric curve in 2-space, what conditions guarantee that the graph will have a horizontal tangent when $t = t_0$?

$y'(t_0) = 0$ and $x'(t_0) \neq 0$

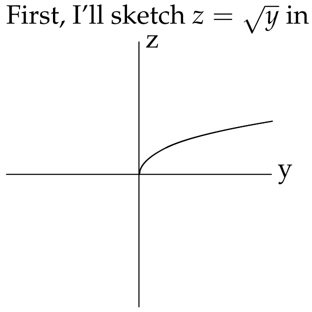
(c) Explain what it means for two non-zero vectors, **u** and **v** , to be parallel. [You can give an equation or a description in words.]

In words: one vector is a (non-zero) scalar multiple of the other. Equivalently, the vectors point in the same or exact opposite directions.

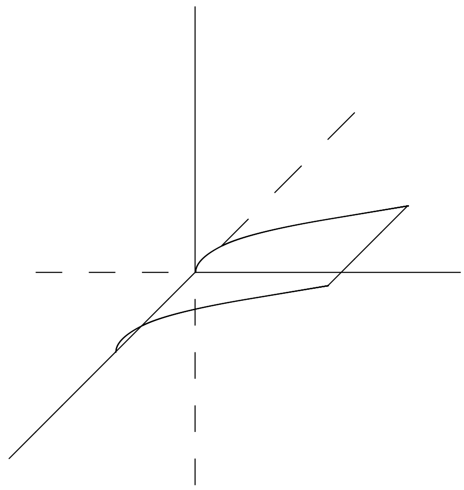
As an equation: $\mathbf{u} = k\mathbf{v}$ for some non-zero scalar k.

6. Sketch $z = \sqrt{y}$ in 3-space.

- First, I'll sketch $z = \sqrt{y}$ in the yz-plane.



- Now for 3-space (extrude parallel to the x-axis):



7. Calculate the work done by the force $\mathbf{F} = \langle 2, -3, 5 \rangle$ acting on an object as it moves from P(2,0,-4) to Q(3, 1,2). Suppose that force is measured in Newtons and distance in meters so that work is given in Newton-meters (Nm).

   $\vec{PQ} = \langle 1, \quad 1, \quad 6 \rangle$

   Work $= \mathbf{F} \bullet \vec{PQ} = \langle 2, \quad -3, \quad 5 \rangle \bullet \langle 1, \quad 1, \quad 6 \rangle = 2 - 3 + 30 = 29\text{Nm}$

8. Determine whether the two lines described below are parallel, intersecting, or skew. If the lines intersect, find the point of intersection.

Line 1	Line 2
$\mathbf{r}(t) = \langle 3, 2, -1 \rangle + t \langle 2, 2, -1 \rangle$	$x = 3 - t$
	$y = 4 + t$
	$z = 3 + 2t$

   - Direction vector for line 1: $\langle 2, \quad 2, \quad -1 \rangle$
     Direction vector for line 2: $\langle -1, \quad 1, \quad 2 \rangle$
     The direction vectors are not parallel (neither is a scalar multiple of the other) so the lines are not parallel. They must intersect or be skew.
   - If the lines intersect, there is a point (x, y,z) on both lines. Thus there must be a t-value, t_1 at which line 1 passes through this point and another (possibly different) t-value, t_2 at which line 2 passes through this point. If such a point exists, the following system of three equations in two unknowns will be consistent:

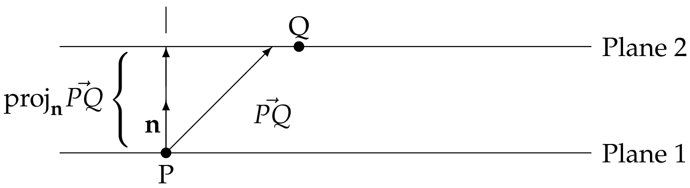
$$
\begin{aligned}
3 + 2t_1 &= 3 - t_2 &\quad& \text{(x-values the same)} \\
2 + 2t_1 &= 4 + t_2 && \text{(y-values the same)} \\
-1 - t_1 &= 3 + 2t_2 && \text{(z-values the same)}
\end{aligned}
$$

On the other hand, if the lines are skew, the above system will be inconsistent.
Solving the first equation for t_2: $t_2 = -2t_1$
Plugging this into the second equation and solving for t_1:
$2 + 2t_1 = 4 + (-2t_1)$
$4t_1 = 2$
$t_1 = \frac{1}{2}$
Thus $t_2 = -2\left(\frac{1}{2}\right) = -1$
Plugging these values into the third equation we get:
$-1 - \frac{1}{2} = 3 - 2$ Simple arithmetic shows this to be false. Thus, the system is inconsistent and so the lines do not intersect; they are skew.

9. Find the distance between the two parallel planes given below:

Plane 1	Plane 2
2x+4y-3z +8=0	4x+8y-6z=4

- First, we pick a point from each plane. There are lots of options. I chose:
  From Plane 1: P(-4, 0,0)
  From Plane 2: Q(1,0,0)
  $\vec{PQ} = \langle 5, \quad 0, \quad 0 \rangle$
- Now we refer to the picture below. The planes are represented as two lines; imagine them extending straight out from the page.



Since the two planes are parallel, they share a normal direction. Selecting our normal vector from the equation for Plane 1, we get: $\mathbf{n} = \langle 2, \quad 4, \quad -3 \rangle$

$$\begin{aligned} \text{proj}_{\mathbf{n}}\vec{PQ} &= \left(\frac{\vec{PQ} \bullet \mathbf{n}}{\mathbf{n} \bullet \mathbf{n}}\right)\mathbf{n} \\ &= \left(\frac{10}{29}\right)\langle 2, \quad 4, \quad -3 \rangle \\ &= \langle \tfrac{20}{29}, \quad \tfrac{40}{29}, \quad \tfrac{-30}{29} \rangle \end{aligned}$$

The distance between the planes is the magnitude of this vector.

$$
\begin{aligned}
||\text{proj}_{\mathbf{n}}\tilde{\mathbf{PQ}}|| &= \frac{10}{29}\,||\mathbf{n}|| \\
&= \frac{10}{29}\sqrt{4+16+9} \\
&= \frac{10\sqrt{29}}{29}
\end{aligned}
$$

10. Find the equation for the plane containing points P(1,0,-2), Q(2,2,2), and R(0,1,3).

    Point: P(1, 0, -2)

    Normal: We find a normal vector by taking the cross product of two vectors parallel to the plane. Two such vectors are:

    $\vec{PQ} = \langle 1, \quad 2, \quad 4 \rangle$

    $\vec{QR} = \langle 2, \quad 1, \quad -1 \rangle$

    $$\mathbf{n} = \vec{PQ} \times \vec{QR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 4 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\mathbf{k}$$

    $= -6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k} = \langle -6, \quad 9, \quad -3 \rangle$

    Plane: Let A = (x, y, z) be an arbitrary point on the plane so that $\vec{PA} = \langle x-1, \quad y, \quad z+2 \rangle$ is parallel to the plane.

    $\mathbf{n} \bullet \vec{PA} = 0$

    $\langle -6, \quad 9, \quad -3 \rangle \bullet \langle x-1, \quad y, \quad z+2 \rangle = 0$

    $\boxed{-6(x-1) + 9y - 3(z+2) = 0}$