

Solutions: Math 3c, Exam 2

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Fall '06

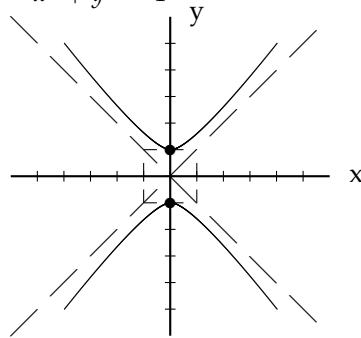
1. Sketch

$$-x^2 + y^2 + \frac{z^2}{4} = 1$$

by first sketching the trace in each of the three coordinate planes and then sketching the surface in 3-space. On the two-dimensional graphs be sure that the location of any vertices is clear and that any asymptotes are drawn in.

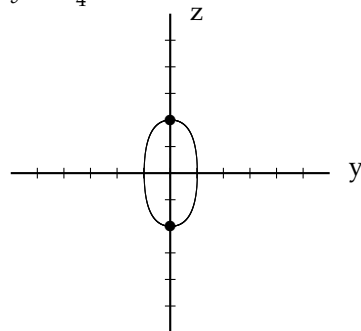
- xy-plane ($z=0$)

$$-x^2 + y^2 = 1$$



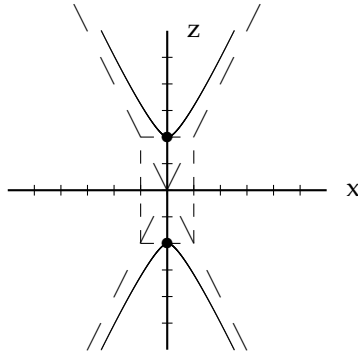
- yz-plane ($x=0$)

$$y^2 + \frac{z^2}{4} = 1$$



- xz-plane ($y=0$)

$$-x^2 + \frac{z^2}{4} = 1$$



- 3-space

Graph is a hyperboloid in one sheet opening along the x-axis. See your textbook if you're not sure what this should look like.

2. Convert the following equation to rectangular coordinates and then sketch the graph.

$$r = 6 \sin \theta$$

Since $y = r \sin \theta$, we'll multiply both sides by r to make the right side equal to $6y$:

$$r(r) = r(6 \sin \theta)$$

$$r^2 = 6r \sin \theta$$

$$(x^2 + y^2) = 6y$$

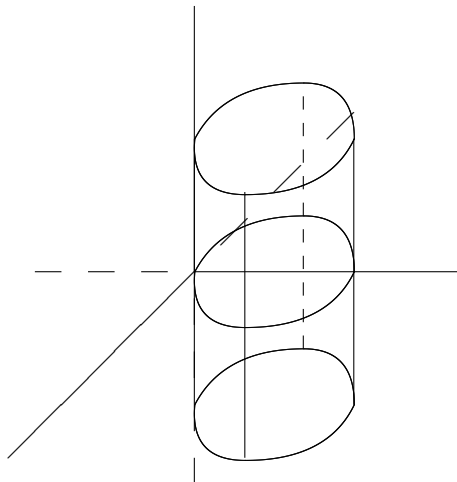
Subtract over the y 's and complete the square:

$$x^2 + y^2 - 6y = 0$$

$$x^2 + (y^2 - 6y + 9) = 9$$

$$x^2 + (y - 3)^2 = 9$$

This is a cylinder: Start with the circle of radius 3 centered at $(0,3)$ and extrude parallel to the z -axis.



3. Let $\mathbf{r}(t) = \langle 2 \sin t - 3, 4 \cos t + 1 \rangle$

(a) Sketch $\mathbf{r}(t)$, including the orientation.

(b) Calculate $\mathbf{r}(\frac{2\pi}{3})$. Sketch this vector in standard position.

$$\mathbf{r}(\frac{2\pi}{3}) = \langle \sqrt{3} - 3, -1 \rangle$$

(c) Calculate $\mathbf{r}'(\frac{2\pi}{3})$. Sketch this vector in translated position, with its tail at the tip of $\mathbf{r}(\frac{2\pi}{3})$.

$$\mathbf{r}'(t) = \langle 2 \cos t, -4 \sin t \rangle$$

$$\mathbf{r}'(\frac{2\pi}{3}) = \langle -1, -2\sqrt{3} \rangle$$

(d) Sketch the tangent line at $t = \frac{2\pi}{3}$.

(e) Find an equation for this tangent line in vector form.

$$\text{starting point: } \mathbf{r}(\frac{2\pi}{3}) = \langle \sqrt{3} - 3, -1 \rangle$$

$$\text{direction vector: } \mathbf{r}'(\frac{2\pi}{3}) = \langle -1, -2\sqrt{3} \rangle$$

$$\text{Line: } L(t) = \langle \sqrt{3} - 3, -1 \rangle + t \langle -1, -2\sqrt{3} \rangle$$

(f) Find an equation for this line by expressing y as a function of x.

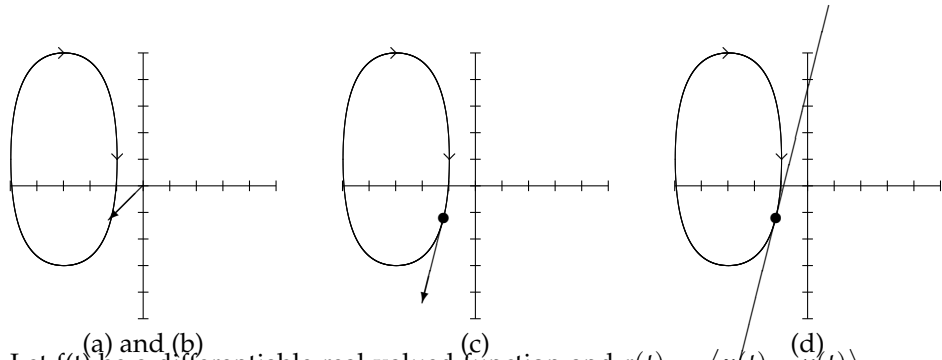
$$\text{point: } (\sqrt{3} - 3, -1)$$

slope: a direction vector of $\langle \underbrace{-1}_{\text{run}}, \underbrace{-2\sqrt{3}}_{\text{rise}} \rangle$ means the slope = $\frac{\text{rise}}{\text{run}} =$

$$\frac{-2\sqrt{3}}{-1} = 2\sqrt{3}$$

$$\text{Line: } y - (-1) = 2\sqrt{3}(x - (\sqrt{3} - 3))$$

$$y = 2\sqrt{3}(x - \sqrt{3} + 3) - 1$$



4. Let $f(t)$ be a differentiable real-valued function and $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ be a differentiable vector-valued function. Prove:

$$\frac{d}{dt} \mathbf{r}(f(t)) = f'(t) \mathbf{r}'(f(t))$$

$$\begin{aligned} \frac{d}{dt} \mathbf{r}(f(t)) &= \frac{d}{dt} \langle x(f(t)), y(f(t)) \rangle \\ &\equiv \left\langle \frac{d}{dt} x(f(t)), \frac{d}{dt} y(f(t)) \right\rangle \end{aligned}$$

evaluate the derivative of vector-valued functions component-wise

$$\equiv \langle x'(f(t))f'(t), y'(f(t))f'(t) \rangle$$

chain rule for real-valued functions of one variable

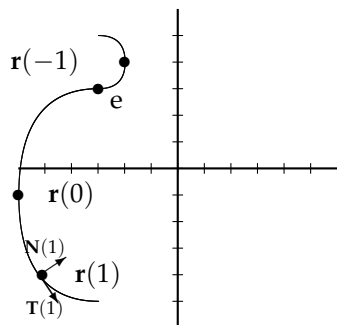
$$\equiv f'(t) \langle x'(f(t)), y'(f(t)) \rangle$$

def. of scalar multiplication

$$\equiv f'(t) \mathbf{r}'(f(t))$$

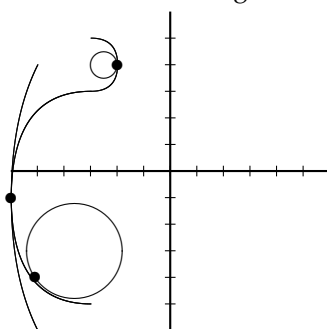
$$\mathbf{r}' = \langle x', y' \rangle$$

5. The following graph shows the graph of a smooth vector-valued function $\mathbf{r}(t)$, with three points labelled. Refer to the graph as you answer the questions that follow. (Each tick mark represents a length of one unit.)



Answers to (d) and (e) are shown on this graph.

- (a) Rank in order from smallest to greatest: $\kappa(-1), \kappa(0), \kappa(1)$.
 $\kappa(0), \kappa(1), \kappa(-1)$
- (b) Sketch the osculating circles at $t = -1, 0$, and 1 .



Only part of the circle at $\mathbf{r}(0)$ is shown, since this is a very large circle.

- (c) Rank in order from smallest to greatest: $\int_{-1}^0 \|\mathbf{r}'(t)\| dt, \int_0^1 \|\mathbf{r}'(t)\| dt,$
 $\int_{-1}^1 \|\mathbf{r}'(t)\| dt$
 $\int_0^1 \|\mathbf{r}'(t)\| dt, \int_{-1}^0 \|\mathbf{r}'(t)\| dt, \int_{-1}^1 \|\mathbf{r}'(t)\| dt$
 (Each integral corresponds to an arc length.)
- (d) Sketch $\mathbf{T}(1)$ and $\mathbf{N}(1)$ in translated position, so that their tails are at $\mathbf{r}(1)$.
- (e) Put an "e" on the curve at the point where \mathbf{N} does not exist.
6. Find the arc length parametrization for $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ with $t=0$ as a reference point.

$$\begin{aligned}
s & \underbrace{=} \int_0^t \|\mathbf{r}'(u)\| \, du \\
& \text{don't forget the} \\
& \text{dummy variable!} \\
& = \int_0^t \sqrt{(-\sin u)^2 + (\cos u)^2 + 1^2} \, du \\
& = \int_0^t \sqrt{\underbrace{\sin^2 u + \cos^2 u}_1 + 1} \, du \\
& = \int_0^t \sqrt{2} \, du \\
& = \sqrt{2}(t - 0) \\
& = \sqrt{2}(t)
\end{aligned}$$

- $s = \sqrt{2}(t) \Rightarrow t = \frac{s}{\sqrt{2}}$.

- Plug into original parametrization: $\mathbf{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$

7. Let $f(x, y) = \cos(x^2y)$

(a) Calculate $\frac{\partial f}{\partial x}$.

$$\frac{\partial}{\partial x} (\cos(x^2y)) = -\sin(x^2y) (2xy) = -2xy \sin(x^2y)$$

(b) Calculate $\frac{\partial^2 f}{\partial y \partial x}$.

$$\begin{aligned}
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (-2xy \sin(x^2y)) \\
& \underbrace{=} -2x \sin(x^2y) + (-2xy) \cos(x^2y) (x^2)
\end{aligned}$$

product rule

$$= \boxed{-2x \sin(x^2y) - 2x^3y \cos(x^2y)}$$

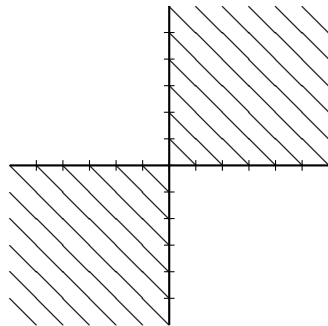
8. Let $f(x, y) = \sqrt{xy}$.

(a) Sketch the domain of this function in 2-space.

$$\text{Domain: } \{(x, y) \mid xy \geq 0\}$$

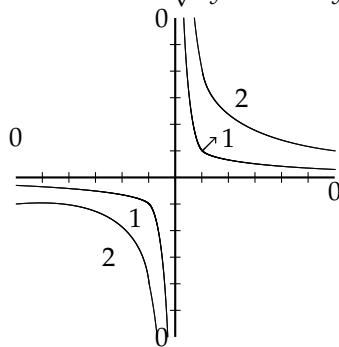
$xy = 0$ along the two coordinate axes.

$xy > 0$ in Quadrants I and III.



(b) Sketch a contour plot for this function, including the level curves $z = 0$, $z = 1$, and $z = 2$.

- $z = 0 \Rightarrow \sqrt{xy} = 0 \Rightarrow xy = 0 \Rightarrow x = 0$
or
 $y = 0$
- $z = 1 \Rightarrow \sqrt{xy} = 1 \Rightarrow xy = 1 \Rightarrow y = \frac{1}{x}$
- $z = 2 \Rightarrow \sqrt{xy} = 2 \Rightarrow xy = 4 \Rightarrow y = \frac{4}{x}$



(c) Calculate $\nabla f(1,1)$.

$$\nabla f = \langle f_x, f_y \rangle = \left\langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \right\rangle$$

$$\nabla f(1,1) = \left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$$

(d) Sketch $\nabla f(1,1)$ on your contour plot so that its tail is at the point $(1,1)$.

Note: This is the (short) on the contour plot above. Notice that $\nabla f(1,1)$ is normal to the level curve $z = 1$ and points in the direction of increasing z -values.

9. Let $f(x,y) = x^2y + x$. Find an equation for the tangent plane to this function at $(1,2)$.

$$f(1,2) = 1^2(2) + 1 = 3$$

$$f_x = 2xy + 1 \qquad f_x(1,2) = 5$$

$$f_y = x^2 \qquad f_y(1,2) = 1$$

$$\boxed{L(x,y) = 3 + 5(x - 1) + (y - 2)}$$

Extra Credit: