

Solutions: 3c Exam 2, Spring 2007

1. Let  $w = f(x, y, z) = x^2 + \frac{y^2}{4} - \frac{z^2}{9}$

- (a) Sketch the level surface  $w = -1$ .
- (b) Sketch the level surface  $w = 0$ .

Rough sketches are ok for both level surfaces; you do not need to draw any 2-dimensional traces unless you find them helpful in sketching your final result.

**Solution is posted separately.**

2. Sketch each surface/region described below in spherical coordinates.

- (a)  $\theta = \pi$
- (b)  $\phi = \frac{\pi}{2}$
- (c)  $\phi \geq \frac{2\pi}{3}$   
 $\rho \leq 5$

**Solution is posted separately.**

3. Let  $\mathbf{r}(t) = \langle 2 \sin t, \cos t \rangle$

- (a) Sketch  $\mathbf{r}(t)$ , including its orientation.  
see graph below

- (b) Calculate  $\mathbf{r}\left(\frac{\pi}{6}\right)$ .

$$\mathbf{r}\left(\frac{\pi}{6}\right) = \left\langle 2 \sin\left(\frac{\pi}{6}\right), \cos\left(\frac{\pi}{6}\right) \right\rangle = \left\langle 2\left(\frac{1}{2}\right), \frac{\sqrt{3}}{2} \right\rangle = \left\langle 1, \frac{\sqrt{3}}{2} \right\rangle$$

- (c) Sketch  $\mathbf{r}\left(\frac{\pi}{6}\right)$  in standard position.  
see graph below

- (d) Calculate  $\mathbf{r}'\left(\frac{\pi}{6}\right)$ .

$$\mathbf{r}'(t) = \langle 2 \cos t, -\sin t \rangle$$

$$\mathbf{r}'\left(\frac{\pi}{6}\right) = \left\langle 2\left(\frac{\sqrt{3}}{2}\right), -\frac{1}{2} \right\rangle = \langle \sqrt{3}, -\frac{1}{2} \rangle$$

- (e) Sketch  $\mathbf{r}'\left(\frac{\pi}{6}\right)$  in translated position with its tail at the tip of  $\mathbf{r}\left(\frac{\pi}{6}\right)$ .  
see graph below

- (f) Sketch the tangent line at  $t = \frac{\pi}{6}$ .  
see graph below

- (g) Find a vector equation for the line in part (f).

**Starting point:**  $\mathbf{r}\left(\frac{\pi}{6}\right) = \left\langle 1, \frac{\sqrt{3}}{2} \right\rangle$

**Direction vector:**  $\mathbf{r}'\left(\frac{\pi}{6}\right) = \langle \sqrt{3}, -\frac{1}{2} \rangle$

**Line:**  $L(t) = \left\langle 1, \frac{\sqrt{3}}{2} \right\rangle + t \langle \sqrt{3}, -\frac{1}{2} \rangle$

(h) Give an equation for the line in part (f) by expressing  $y$  as a function of  $x$ .

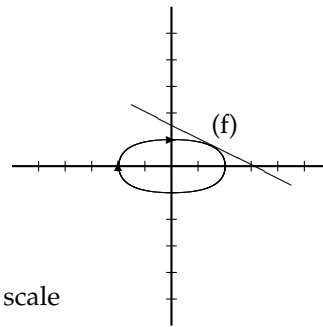
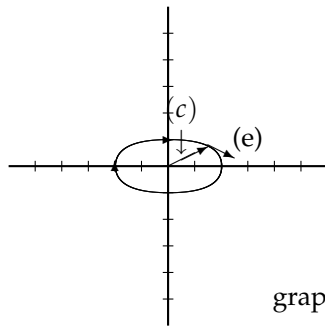
**Point:**  $\left(1, \frac{\sqrt{3}}{2}\right)$

**Slope:**  $\left\langle \underbrace{\sqrt{3}}_{\text{run}}, \underbrace{-\frac{1}{2}}_{\text{rise}} \right\rangle$

$$\frac{\text{rise}}{\text{run}} = \frac{-\frac{1}{2}}{\sqrt{3}} = -\frac{1}{2\sqrt{3}}$$

**Line:**  $y - \underbrace{\frac{\sqrt{3}}{2}}_{\text{rise}} = \underbrace{-\frac{1}{2\sqrt{3}}}_{\text{slope}} \left( \underbrace{x - 1}_{\text{run}} \right)$

$$y = \frac{\sqrt{3}}{2} - \frac{1}{2\sqrt{3}}(x - 1)$$



graphs are not to scale

4. Let  $\mathbf{r}(t) = \left\langle \frac{\sqrt{6}}{3}t^3, t, \frac{3}{2}t^2 \right\rangle$  be a function representing an object's position,  $\mathbf{r}$ , as a function of time,  $t$ .

(a) Find the object's displacement over the time interval  $[0,2]$ .

$$\begin{aligned} \text{displacement} &= \mathbf{r}(2) - \mathbf{r}(0) \\ &= \left\langle \frac{8\sqrt{6}}{3}, 2, 6 \right\rangle - \langle 0, 0, 0 \rangle \\ &= \left\langle \frac{8\sqrt{6}}{3}, 2, 6 \right\rangle \end{aligned}$$

(b) Find the distance traveled by the object over the time interval  $[0,2]$ . (Set up, but don't evaluate the appropriate interval.)

$$\mathbf{r}'(t) = \langle \sqrt{6}t^2, 1, 3t \rangle$$

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{(\sqrt{6}t^2)^2 + (1)^2 + (3t)^2} \\ &= \sqrt{6t^4 + 1 + 9t^2} \end{aligned}$$

$$\begin{aligned} \text{distance} &= \int_0^2 \|\mathbf{r}'(t)\| dt \\ &= \int_0^2 \sqrt{6t^4 + 9t^2 + 1} dt \end{aligned}$$

5. Let  $\mathbf{r}_1(t) = \langle x_1(t), y_1(t) \rangle$  and  $\mathbf{r}_2(t) = \langle x_2(t), y_2(t) \rangle$  be two differentiable vector-valued functions. Prove:

$$\frac{d}{dt} (\mathbf{r}_1(t) \bullet \mathbf{r}_2(t)) = \mathbf{r}'_1(t) \bullet \mathbf{r}_2(t) + \mathbf{r}_1(t) \bullet \mathbf{r}'_2(t)$$

**LHS:** (left-hand side)

$$\begin{aligned} \frac{d}{dt} (\mathbf{r}_1(t) \bullet \mathbf{r}_2(t)) &= \frac{d}{dt} (\langle x_1(t), y_1(t) \rangle \bullet \langle x_2(t), y_2(t) \rangle) \\ &\stackrel{\text{def. of dot product}}{=} \frac{d}{dt} (x_1(t)x_2(t) + y_1(t)y_2(t)) \\ &\stackrel{\text{product rule for real-valued functions of one variable}}{=} \underbrace{x'_1(t)x_2(t) + x_1(t)x'_2(t)}_{\frac{d}{dt}(x_1(t)x_2(t))} + \underbrace{y'_1(t)y_2(t) + y_1(t)y'_2(t)}_{\frac{d}{dt}(y_1(t)y_2(t))} \end{aligned}$$

**RHS:** (right-hand side)

$$\begin{aligned}
& \mathbf{r}'_1(t) \bullet \mathbf{r}_2(t) + \mathbf{r}_1(t) \bullet \mathbf{r}'_2(t) \\
&= \langle x'_1(t), y'_1(t) \rangle \bullet \langle x_2(t), y_2(t) \rangle + \langle x_1(t), y_1(t) \rangle \bullet \langle x'_2(t), y'_2(t) \rangle \\
&\equiv \underbrace{x'_1(t)x_2(t) + y'_1(t)y_2(t) + x_1(t)x'_2(t) + y_1(t)y'_2(t)}_{\text{def. of dot product}} \\
&\equiv \underbrace{x'_1(t)x_2(t) + x_1(t)x'_2(t) + y'_1(t)y_2(t) + y_1(t)y'_2(t)}_{\text{rearranging terms}}
\end{aligned}$$

**LHS-RHS** Since the left-hand side is equal to the right-hand side, the equation is proved.

6. The graph of a smooth curve  $\mathbf{r}(t)$  is shown below with several points marked off.

(a) Sketch **T** and **N** at point  $\mathbf{r}(1)$ , assuming that each tick mark on the graph represents a length of one unit.

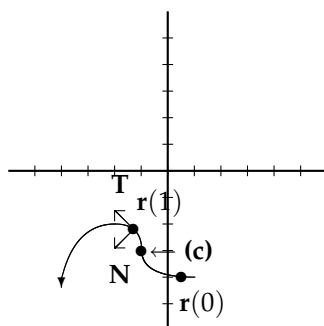
(b) Which is greater,  $\kappa(0)$  or  $\kappa(1)$ ?

$\kappa(1)$  is greater since the curvature is greater here. (Note that this means the radius of curvature is smaller.)

(c) Put a "c" on the graph at the point where there is no unit normal vector.

(d) Suppose that  $\|\mathbf{r}'(t)\|$  is constant; is it smaller than, equal to, or greater than 1?

Greater than 1; the interval  $[0,1]$  is mapped to a curve whose arc length is greater than 1, so the interval  $[0,1]$  is "stretched" by this mapping.



7. Suppose an object's acceleration is given as a function of time by the equation  $\mathbf{a}(t) = \langle e^t, \cos 2t \rangle$  and that the object's initial velocity (when

time  $t=0$  is  $\langle 3, -2 \rangle$ . Find a formula for the object's velocity as a function of time.

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t) dt \\ &= \int \langle e^t, \cos 2t \rangle dt \\ &= \langle e^t, \frac{1}{2} \sin 2t \rangle + \mathbf{c}\end{aligned}$$

$$\begin{aligned}\mathbf{v}(0) &= \langle e^0, \frac{1}{2} \sin(2(0)) \rangle + \mathbf{c} \\ \langle 3, -2 \rangle &= \langle 1, 0 \rangle + \mathbf{c} \\ \langle 3, -2 \rangle - \langle 1, 0 \rangle &= \mathbf{c} \\ \langle 2, -2 \rangle &= \mathbf{c}\end{aligned}$$

$$\begin{aligned}\mathbf{v}(t) &= \langle e^t, \frac{1}{2} \sin(2t) \rangle + \langle 2, -2 \rangle \\ &= \langle e^t + 2, \frac{1}{2} \sin(2t) - 2 \rangle\end{aligned}$$

8. Let  $f(x, y) = x^2 + y^2$ .

(a) Find an equation for  $L(x, y)$ , the tangent plane to the graph of  $f$  at  $(1, 1)$ .

$$\begin{aligned}f(1, 1) &= 1^2 + 1^2 = 2 \\ f_x(x, y) &= 2x & f_x(1, 1) &= 2 \\ f_y(x, y) &= 2y & f_y(1, 1) &= 2\end{aligned}$$

$$\begin{aligned}L(x, y) &= f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) \\ &= 2 + 2(x - 1) + 2(y - 1)\end{aligned}$$

(b) Find a formula for  $E(x, y)$  at this point.

$$\begin{aligned}E(x, y) &= f(x, y) - L(x, y) \\ &= x^2 + y^2 - [2 + 2(x - 1) + 2(y - 1)] \\ &= x^2 + y^2 - 2 - 2(x - 1) - 2(y - 1)\end{aligned}$$

(c) Find a formula for  $d(P, P_0)$  if  $P$  is an arbitrary point  $(x, y)$  and  $P_0 = (1, 1)$ .

$$d(P, P_0) = \sqrt{(x - 1)^2 + (y - 1)^2}$$

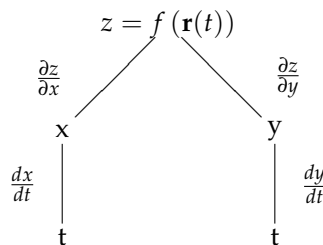
(d) **(Extra credit: 2 points)** Prove that  $f$  is differentiable at  $P_0 = (1, 1)$  by

showing that  $\lim_{P \rightarrow P_0} \frac{E(x,y)}{d(P,P_0)} = 0$ .

$$\begin{aligned}
 \lim_{P \rightarrow P_0} \frac{E(x,y)}{d(P,P_0)} &= \lim_{P \rightarrow P_0} \frac{x^2 + y^2 - 2 - 2(x-1) - 2(y-1)}{\sqrt{(x-1)^2 + (y-1)^2}} \\
 &= \lim_{P \rightarrow P_0} \frac{x^2 + y^2 - 2 - 2x + 2 - 2y + 2}{\sqrt{(x-1)^2 + (y-1)^2}} \\
 &= \lim_{P \rightarrow P_0} \frac{x^2 + y^2 - 2x - 2y + \overbrace{2}^{1+1}}{\sqrt{(x-1)^2 + (y-1)^2}} \\
 &= \lim_{P \rightarrow P_0} \frac{x^2 - 2x + 1 + y^2 - 2y + 1}{\sqrt{(x-1)^2 + (y-1)^2}} \\
 &= \lim_{P \rightarrow P_0} \frac{(x-1)^2 + (y-1)^2}{\sqrt{(x-1)^2 + (y-1)^2}} \\
 &= \lim_{P \rightarrow P_0} \sqrt{(x-1)^2 + (y-1)^2} \\
 &= \sqrt{(1-1)^2 + (1-1)^2} \\
 &= 0
 \end{aligned}$$

9. Let  $\mathbf{r}(t) = \langle t^2, \sin t \rangle$  and let  $f(x, y) = x^2y$ .

Use the chain rule to calculate  $\frac{d}{dt}f(\mathbf{r}(t))$ . [Note: to receive full credit you *must* use the chain rule to calculate your answer.]



$$\begin{aligned}
 \frac{d}{dt}f(\mathbf{r}(t)) &= \frac{dz}{dt} \\
 &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\
 &= (2xy)(2t) + (x^2)(\cos t) \\
 &= (2t^2 \sin t)(2t) + (t^2)^2 (\cos t) \\
 &= 4t^3 \sin t + t^4 \cos t
 \end{aligned}$$