Solutions: Math 3c, Midterm 3, Fall '06

1. Find the tangent plane to the surface $\sin(xy) + z = -1$ at the point (0,1,-1).

   Point: $(0, 1, -1)$

   Normal vector: Letting f(x,y,z) = $\sin(xy) + z)$, we recognize the surface as a level surface of the function f. The gradient will be normal to the surface, and hence to its tangent plane:
   $\bigtriangledown f(0, 1, -1)$:
   $$\bigtriangledown f = \langle f_x, \quad f_y, \quad f_z \rangle = \langle y\cos(xy), \quad x\cos(xy), \quad 1 \rangle$$
   $$\bigtriangledown f(0, 1, -1) = \langle 1, \quad 0, \quad 1 \rangle$$

   Plane:
   $\langle 1, \quad 0, \quad 1 \rangle \bullet \langle x - 0, \quad y - 1, \quad z + 1 \rangle = 0$
   $\boxed{x + z + 1 = 0}$

2. Let $f(x, y) = x^3 + y^3$ and let D = $\{(x, y) \,|x^2 + y^2 \le 1\}$ be the solid unit disk in the xy-plane.

   (a) We know that f achieves an absolute max and min on D by the Extreme Value theorem, which applies here since f is continuous and D is closed and bounded. The max and min could occur at one of two types of locations: critical points or boundary points.

   (b) Find the location and value of the absolute max and min. Use the method of Lagrange multipliers on the boundary.

   Critical points: $\bigtriangledown f = \langle 3x^2, \quad 3y^2 \rangle$ Since $\bigtriangledown f$ is always defined, the critical points will be where $\bigtriangledown f = \mathbf{0}$

   $$\begin{aligned} \bigtriangledown f = \mathbf{0} \quad &\Rightarrow \quad \langle 3x^2, \quad 3y^2 \rangle = \langle 0, \quad 0 \rangle \\ &\Rightarrow \\ 3x^2 = 0 \quad &\mathit{and} \quad 3y^2 = 0 \\ &\Rightarrow \\ x = 0 \quad &\mathit{and} \quad y = 0 \end{aligned}$$

   We have one critical point: (0,0).

   Boundary points: The constraint curve is the unit circle, defined by $g(x, y) = x^2 + y^2 = 1$.
   $\bigtriangledown g = \langle 2x, \quad 2y \rangle$.

   $$\begin{aligned} \bigtriangledown f = \lambda \bigtriangledown g \quad &\Rightarrow \quad \langle 3x^2, \quad 3y^2 \rangle = \lambda \langle 2x, \quad 2y \rangle \\ &\Rightarrow \quad 3x^2 = \lambda(2x) \\ &\qquad \mathit{and} \\ &\qquad 3y^2 = \lambda(2y) \end{aligned}$$

Thus we have the following system of 3 equations in 3 unknowns:
(1) $3x^2 = \lambda(2x)$
(2) $3y^2 = \lambda(2y)$
(3) $x^2 + y^2 = 1$ (constraint equation)
From (1):
$3x^2 = \lambda(2x) \Rightarrow \lambda = \frac{3x}{2}$ *or* $x = 0$
From (2):
$3y^2 = \lambda(2y) \Rightarrow \lambda = \frac{3y}{2}$ *or* $y = 0$

Case 1: $x \neq 0$ **and** $y \neq 0$ Then $\lambda = \frac{3x}{2} = \frac{3y}{2} \Rightarrow x = y$

Plugging into the constraint equation, we get:

$$\begin{aligned} x^2 + y^2 = 1 &\Rightarrow x^2 + x^2 = 1 \\ &\Rightarrow 2x^2 = 1 \\ &\Rightarrow x^2 = \frac{1}{2} \\ &\Rightarrow x = \pm\frac{1}{\sqrt{2}} \end{aligned}$$

Combining this with the fact that x=y, we have two potential locations for the max and min:
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^3 + \left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$
Simlilarly, $f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$.

Case 2: x = 0 Equation (1) is satisfied, as it becomes 0=0.
From (3): $0^2 + y^2 = 1 \Rightarrow y = \pm 1$
Then (2) becomes $3 = \lambda(2)(\pm 1)$
This is satisfied if $\lambda = \frac{3}{2}$ when y = 1 and $\lambda = -\frac{3}{2}$ when $y = -1$.
Thus we have two additional points that are potential locations for the max and min: $(0, 1)$ and $(0, -1)$.
$f(0, 1) = 0^3 + 1^3 = 0 + 1 = 1$
$f(0, -1) = 0^3 + (-1)^3 = 0 + -1 = -1$

Case 3: y = 0 Equation (2) is satisfied as it becomes 0=0.
From (3): $x^2 + 0 = 1 \Rightarrow x = \pm 1$.
Then (1) becomes $3 = \lambda(2)(\pm 1)$ This is satisfied if $\lambda = \frac{3}{2}$ when x = 1 and $\lambda = -\frac{3}{2}$ when $x = -1$.
Thus we have two additional points that are potential locations for the max and min: $(1, 0)$ and $(-1, 0)$.
$f(1, 0) = 1^3 + 0^3 = 1 + 0 = 1$
$f(-1, 0) = (-1)^3 + 0^3 = -1 + 0 = -1$

	(x,y)	f(x,y)
critical points	$\{(0,0)$	0
boundary points	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$
	$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{1}{\sqrt{2}}$
	$(0,1)$	1
	$0,-1)$	-1
	$(1,0)$	1
	$(-1,0)$	-1

The max is 1 and this occurs at (1,0) and (0,1).
The min is -1 and this occurs at $(-1,0)$ and $(0,-1)$

3. (a) Fill in the blank. Let z=f(x,y) be a continuous function such that all of the second partials are also continuous. If f has a relative max or min, this must occur at a critical point, which is a point where $\underline{\nabla f}$ = **0** (the zero vector.) To determine whether we have a max or min at such a point, we can use the second partials test. We define

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

If D = _0_, the test is inconclusive and we know NOTHING!

(b) According to the second partials test, what must be true at a critical point for us to conclude that we have a relative max here?
$D > 0$ and $f_{xx} < 0$.

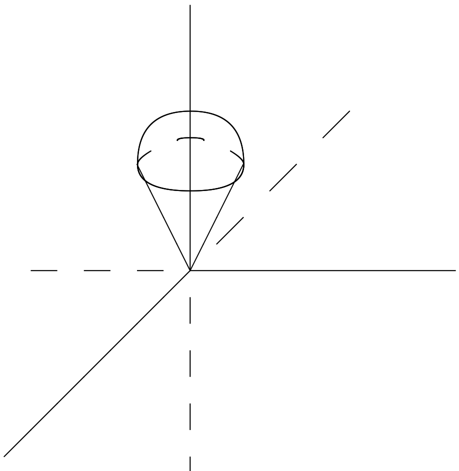
(c) According to the second partials test, what must be true at a critical point for us to conclude that we have a relative min here?
$D > 0$ and $f_{xx} > 0$.

(d) According to the second partials test, what must be true at a critical point for us to conclude that we have a saddle point here?
$D < 0$.

4. Let G be the solid enclosed by the cone $z = \sqrt{x^2+y^2}$ and the hemisphere $z = \sqrt{4-x^2-y^2}$. Set up, but DO NOT EVALUATE a triple integral that will allow us to calculate the volume of G:

(a) in rectangular coordinates
(b) in cylindrical coordinates
(c) in spherical coordinates.

Note that this is an " ice cream cone with ice cream in it;" the bottom of this region is the cone, the top is part of the upper hemisphere of radius 2, centered at the origin. The picture below is not to scale.

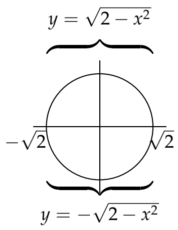


For both rectangular and cylindrical coordinates, we will need to find R, the projection of this solid onto the xy-plane. This will be the region enclosed by the curve of intersection of the two surfaces; we find this by setting the two expressions for z equal to each other.

$$\begin{aligned} \sqrt{x^2+y^2} = \sqrt{4-x^2-y^2} \quad &\Rightarrow \quad x^2+y^2 = 4-x^2+y^2 \\ &\Rightarrow \quad 2x^2+2y^2 = 4 \\ &\Rightarrow \quad x^2+y^2 = 2 \end{aligned}$$

This is the circle of radius $\sqrt{2}$ centered at the origin.

Rectangular coordinates:



We can view R as a type I region: x varies from $-\sqrt{2} to \sqrt{2}$.
y varies from bottom = -$\sqrt{2-x^2}$ to top = $\sqrt{2-x^2}$.

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 1 \; dzdydx$$

Cylindrical coordinates:

The bottom is still the cone; in cylindrical coordinates this is $z = r$.

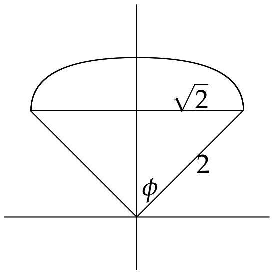
The top is still the hemisphere; in cylindrical coordinates this is $z = \sqrt{4 - r^2}$.

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r \, dzdrd\theta$$

Spherical coordinates:

Sweeping out the cone, ρ varies from 0 to 2 (the radius of the hemisphere.)

Recalling that in spherical coordinates, a cone is defined by a constant ϕ function, we need to determine what value of ϕ corresponds to this cone. Looking at a cross-section in the yz-plane:



$\sin\phi = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$ since, by inspection, ϕ is an angle between 0 and $\frac{\pi}{2}$.

Thus the cone is the constant function $\phi = \frac{\pi}{4}$, and so in sweeping out the cone, ϕ must vary from 0 to $\frac{\pi}{4}$.

In order to sweep out the entire cone, θ must vary from 0 to 2π.

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2} \rho^2 \sin\phi \, d\rho d\phi d\theta$$

5. Consider the parametric surface defined by $\mathbf{r}(u,v) = \langle 2u, u^2, uv \rangle$. Find the surface area over the region in the uv-plane defined by $\begin{matrix} 0 \le u \le \sqrt{3} \\ 0 \le v \le 1 \end{matrix}$.

$\frac{\partial \mathbf{r}}{\partial u} = \langle 2, \quad 2u, \quad v \rangle$
$\frac{\partial \mathbf{r}}{\partial v} = \langle 0, \quad 0, \quad u \rangle$

$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} =$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2u & v \\ 0 & 0 & u \end{vmatrix} = \begin{vmatrix} 2u & v \\ 0 & u \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & v \\ 0 & u \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 2u \\ 0 & 0 \end{vmatrix} \mathbf{k} =$$

$2u^2\mathbf{i} - 2u\mathbf{j} = \langle 2u^2, \quad -2u, \quad 0 \rangle$.

$$\left|\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|\right| = \sqrt{(2u^2)^2 + (-2u)^2 + 0^2} = \sqrt{4u^4 + 4u^2}$$

$$\begin{aligned}
Surface\ area &= \int_0^1 \int_0^{\sqrt{3}} \sqrt{4u^4 + 4u^2}\, dudv \\
&= \int_0^1 \int_0^{\sqrt{3}} \sqrt{4u^2\left(u^2+1\right)}\, dudv \\
&= \int_0^1 \int_0^{\sqrt{3}} \sqrt{4u^2}\sqrt{u^2+1}\, dudv \\
&\underbrace{=}_{\substack{\sqrt{4u^2} = 2|u| = \\ 2u \text{ since} \\ 0 \le u \le \sqrt{3}}} \int_0^1 \int_0^{\sqrt{3}} 2u\sqrt{u^2+1}\, dudv \\
&\underbrace{=}_{\substack{w = u^2+1 \\ w' = 2u \\ dw = 2udu}} \int_0^1 \int_1^4 w^{\frac{1}{2}}\, dwdv \\
&= \int_0^1 \left.\frac{2}{3}w^{\frac{3}{2}}\right|_1^4 dv \\
&= \int_0^1 \frac{2}{3}(8-1)\ dv \\
&= \int_0^1 \frac{14}{3}\ dv \\
&= \frac{14}{3}(1) = \frac{14}{3}
\end{aligned}$$

6. Find the volume of the region enclosed by the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$. [Hint: Let S be the solid unit sphere in the u,v,w-coordinate system. Map S to the region enclosed by the ellipsoid and integrate over S.]

   The unit sphere in the u,v,w system is centered at the origin. Along the three coordinate axes, the sphere extends 1 unit away in every direction: forward, back, left, right, up, and down.

   The ellipsoid in the x,y,z system is centered at the origin. Along the three coordinate axes it extends 1 unit forward and back, 2 units left and right, and 3 units up and down. Thus we can view the ellipsoid as a sphere that has been stretched by a factor of 2 along the y-axis, and a factor of 3 along the z-axis. We accomplish this with the following transformation:

$$T(u, v, w) = (u, 2v, 3w)$$

Equivalently

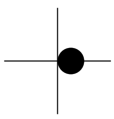
$$x = u$$
$$y = 2v$$
$$z = 3w$$

$$Volume = \iiint\limits_G 1 \, dV_{x,y,z} = \iiint\limits_S 1 \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \, dV_{u,v,w}$$

- $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$
  $= \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} 1 - \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} 0 + \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} 0 = 1(6) + 0 + 0 = 6$
- $\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = |6| = 6$

$$\begin{aligned} Volume &= \iiint\limits_S 1(6) \, dV_{u,v,w} \\ &= 6 \iiint\limits_S 1 \, dV_{u,v,w} \\ &= 6(volume\ of\ S) \\ &= 6\left(\frac{4}{3}\pi\right) \\ &= \frac{24\pi}{3} \\ &= 8\pi \end{aligned}$$

7. Consider the circular lamina of radius 1 shown below. Suppose that its density function is $\delta(x, y) = y + 1$.



Calculate the mass of the lamina.

Note that the circle of *diameter* 1 in this position would be given by the polar equation $r = \cos\theta$. The circle of *radius* 1 is twice as big; it is given by the equation $r = 2\cos\theta$.

$$\begin{aligned}
Mass &= \iint_R \delta(x,y)\, dA \\
&= \iint_R (y+1)\, dA \\
&\underbrace{=}_{\text{switch to polar coordinates}} \int_0^{\pi} \int_0^{2\cos\theta} r\,(r\sin\theta + 1)\ dr d\theta \\
&= \int_0^{\pi} \int_0^{2\cos\theta} \left(r^2 \sin\theta + r\right)\ dr d\theta \\
&= \int_0^{\pi} \left(\frac{r^3}{3}\sin\theta + \frac{r^2}{2}\right)\Bigg|_0^{2\cos\theta} d\theta \\
&= \int_0^{\pi} \left(\frac{8\cos^3\theta}{3}\sin\theta + \frac{4\cos^2\theta}{2}\right)\ d\theta \\
&= \underbrace{\int_0^{\pi} \left(\frac{8}{3}\cos^3\theta\right)\sin\theta d\theta}_{A} + \underbrace{\int_0^{\pi} 2\cos^2\theta\ d\theta}_{B}
\end{aligned}$$

A:

$$\begin{aligned}
\int_0^{\pi} \left(\frac{8}{3}\cos^3\theta\right)\sin\theta d\theta &\underbrace{=}_{\substack{u=\cos\theta \\ u'=-\sin\theta \\ du=-\sin\theta\, d\theta}} -\int_1^{-1} \left(\frac{8}{3}\right) u^3\ du \\
&= -\left(\frac{8}{3}\right)\left(\frac{u^4}{4}\right)\Bigg|_1^{-1} \\
&= -\left(\frac{8}{3}\right)\left(\frac{1}{4} - \frac{1}{4}\right) \\
&= 0
\end{aligned}$$

B:

$$\int_0^{\pi} 2\cos^2\theta \, d\theta \underbrace{=}_{\text{double angle formula}} \int_0^{\pi} (1+\cos 2\theta)\, d\theta$$

$$\underbrace{=}_{\substack{w=2\theta \\ w'=2 \\ dw=2d\theta}} \frac{1}{2}\int_0^{2\pi} (1+\cos w)\, dw$$

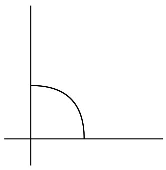
$$= \frac{1}{2}\,(w+\sin w)|_0^{2\pi}$$

$$= \left(\frac{1}{2}\right)(2\pi)$$

$$= \pi$$

Mass = A + B = 0 + $\pi = \pi$.

8. Let $\mathbf{F}(x,y) = \langle xy, y\rangle$ be a force field. Let C be the quarter of the circle of radius 2, shown below, with counterclockwise orientation. Find the work done by $\mathbf{F}$ on a particle travelling along C.



Parametrize C: $\mathbf{r}(t) = \langle 2\cos t, \quad 2\sin t\rangle\,;\ 0 \le t \le \frac{\pi}{2}$

$\mathbf{r}'(t) = \langle -2\sin t, \quad 2\cos t\rangle$

Along the curve, $\mathbf{F}$ can be expressed in terms of t as follows:

$\mathbf{F}(\mathbf{r}(t)) = \langle (2\cos t)(2\sin t), \quad 2\sin t\rangle = \langle 4\cos t\sin t, \quad 2\sin t\rangle$

$$
\begin{aligned}
Work &= \int_C \mathbf{F} \bullet \mathbf{T}\, ds \\
&= \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r}(t) \bullet \mathbf{r}'(t)\, dt \\
&= \int_0^{\frac{\pi}{2}} \left\langle 4\cos t \sin t, \quad 2\sin t \right\rangle \bullet \left\langle -2\sin t, \quad 2\cos t \right\rangle\, dt \\
&= \int_0^{\frac{\pi}{2}} \left(-8\sin^2 t \cos t + 4\cos t \sin t \right)\, dt \\
&= \int_0^{\frac{\pi}{2}} \left(-8\sin^2 t + 4\sin t \right) \cos t\, dt \\
&\underbrace{=}_{\substack{u = \sin t \\ u' = \cos t \\ du = \cos t\, dt}} \int_0^1 \left(-8u^2 + 4u \right)\, du \\
&= \left(-\frac{8u^3}{3} + 2u^2 \right)\Bigg|_0^1 \\
&= \left[\left(\frac{-8}{3} + 2 \right) - 0 \right] \\
&= \frac{-2}{3}
\end{aligned}
$$