Solutions: Math 3c, Exam 3

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Spring 2007

1. Consider the surface defined by the equation

   $$\underbrace{x^2 - 4y^2 + 2z}_{f(x,y,z)} = 5$$

   Find the equation of the tangent plane to this surface at the point (1, 0, 2).

   Point: (1,0,2)

   Normal: $\mathbf{N} = \bigtriangledown f(1,0,2)$
   $\bigtriangledown f = \langle 2x, \quad -8y, \quad 2\rangle$
   $\bigtriangledown f(1,0,2) = \langle 2, \quad 0, \quad 2\rangle$

   Plane: $\langle 2, \quad 0, \quad 2\rangle \bullet \langle x-1, \quad y, \quad z-2\rangle = 0$
   $2(x-1) + 2(z-2) = 0$

2. Let f(x,y) $=x^2 + y^3$ and let $D = \{(x,y)\,|x^2+y^2 \le 1\}$ be the unit disk in the xy-plane.

   (a) The Extreme Value theorem ensures that f achieves an absolute max and an absolute min on D, since f is continuous and D is closed and bounded. The max and min could occur at two types of locations: critcal points and boundary points.

   (b) Find the location and value of the absolute max and absolute min of f on D. Use the method of Lagrange multipliers on the boundary.

   Critical points: $\bigtriangledown f = \langle f_x, \quad f_y\rangle = \langle 2x, \quad 3y^2\rangle$
   $\bigtriangledown f$ is always defined.
   $\bigtriangledown f = \mathbf{0} \Rightarrow \underbrace{2x = 0}_{x=0} \; and \; \underbrace{3y^2 = 0}_{y=0}$
   We have one critical point: (0,0).

   Boundary points: The boundary is the set of points satisfying the *equation* $\underbrace{x^2 + y^2}_{g(x,y)} = 1$
   $\bigtriangledown g = \langle 2x, \quad 2y\rangle$, which is not **0** on the boundary.

$$\bigtriangledown f = \lambda \bigtriangledown g \Rightarrow \langle 2x, \quad 3y^2 \rangle = \langle 2x, \quad 2y \rangle \Rightarrow \begin{cases} 2x = \lambda(2x) \\ \quad and \\ 3y^2 = \lambda(2y) \end{cases}$$

Thus we have the following system of 3 equations in 3 unknowns (x, y, and λ).

(1) $\quad 2x = \lambda(2x) \Rightarrow 1 = \lambda \ or \ x = 0$

(2) $\quad 3y^2 = \lambda(2y) \Rightarrow \frac{3y}{2} = \lambda \ or \ y = 0$

(3) $\quad x^2 + y^2 = 1$

Case 1: (the nice case) $x \neq 0, y \neq 0$

If neither variable is 0, we are able to solve (1) and (2) for λ, and we can set the two expressions for λ equal to each other.

$$\begin{aligned} \lambda &= \lambda \\ 1 &= \tfrac{3y}{2} \\ \tfrac{2}{3} &= y \end{aligned}$$

Plugging this result into (3):

$$\begin{aligned} x^2 + \left(\frac{2}{3}\right)^2 = 1 \quad &\Rightarrow \quad x^2 + \frac{4}{9} = 1 \\ &\Rightarrow \quad x^2 = \frac{5}{9} \\ &\Rightarrow \quad x = \pm\frac{\sqrt{5}}{3} \end{aligned}$$

From case 1, we get 2 points: $\left(\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$ and $\left(-\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$

Case 2: x = 0

(1) becomes 0=0, which is always true.

Plugging into (3), we get $0^2 + y^2 = 1 \Rightarrow y = \pm 1$.

Plugging into (2):

$x = 0, \ y = 1 \Rightarrow \lambda = \frac{3}{2} \quad x = 0, \ y = -1 \Rightarrow \lambda = -\frac{3}{2}$

Thus case 2 gives us 2 more points: (0,1) and $(0, -1)$.

Case 3: y = 0 (2) becomes 0=0, which is always true.

Plugging into (3), we get $x^2 + 0^2 = 1 \Rightarrow x = \pm 1$

Plugging into (1):

$x = 1, y = 0 \Rightarrow \lambda = 1 \quad x = -1, y = 0 \Rightarrow \lambda = 1$

Thus case 3 gives us two more points: (1,0) and $(-1, 0)$.

	(x,y)	f(x,y)
critical point	$(0,0)$	0
boundary points	$\left(\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$	$\frac{23}{27}$
	$\left(-\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$	$\frac{23}{27}$
	$(0,1)$	$1 \leftarrow max$
	$(0,-1)$	$-1 \leftarrow min$
	$(1,0)$	$1 \leftarrow max$
	$(-1,0)$	$1 \leftarrow max$

The max is 1; it occurs at (0,1), (1,0), and $(-1,0)$.
The min is -1; it occurs at $(0,-1)$.

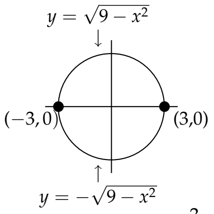
3. Let G be the solid bounded by the surfaces $z = x^2 + y^2$ and z=9. Write down, but don't evaluate, a triple integral that could be used to calculate the volume of G as an iterated integral:

   (a) in rectangular coordinates

   bottom: $z = x^2 + y^2$ (elliptic paraboloid)

   top: z = 0 (plane)

   projection onto xy-plane: To find the projection set the equations for the top and bottom equal to each other; this will be a curve whose interior (in the xy-plane) is the projection.
   $x^2 + y^2 = 9$
   This is a circle of radius 3, centered at the origin.

   Volume = $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x^2+y^2}^{9} 1 \, dzdydx$

   (b) in cylindrical coordinates

   bottom: $z = x^2 + y^2 \Rightarrow z = r^2$

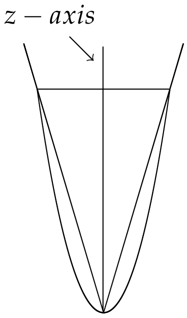
   Top: z= 9

   Projection onto xy-plane: Again, we have a disc of radius 3, centered at the origin.

   Volume = $\int_{0}^{2\pi} \int_{0}^{3} \int_{r^2}^{9} r \, dzdrd\theta$

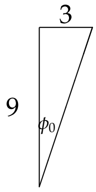
   (c) in spherical coordinates.

   This one's a bit trickier. Let's examine the cross-section of this region in the yz-plane. Note that we would get the same cross-section in any plane perpendicular to the xy-plane; such a plane is actually the union of two half-planes, in this case corresponding to the constant θ functions $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ The cross-section is the lower part of the parabola $z = y^2$ as y ranges from $-3to3$, with a "ceiling" at a height of 9:



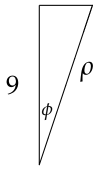
The diagonal lines above show that for small ϕ values (inside the two diagonal lines) ρ will vary from 0 to its value on the "ceiling". For large ϕ values (outside the two diagonal lines, but no more than $\frac{\pi}{2}$) ρ will vary from 0 to its value on the parabola. So we'll have to break this up into two integrals.

Finding the boundary ϕ value (call it ϕ_0): Using the triangle



we see that $\tan\phi_0 = \frac{3}{9} = \frac{1}{3}$ So $\phi_0 = \arctan\frac{1}{3}$

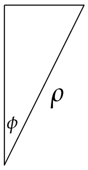
Finding the upper ρ limit if $\phi <$ arctan $\frac{1}{3}$: For small ϕ values, we have the following picture:



$$\cos\phi = \frac{9}{\rho} \Rightarrow \rho = \frac{9}{\cos\phi} = 9\sec\phi$$

Finding the upper ρ limit if $\arctan\frac{1}{3} < \phi < \frac{\pi}{2}$**:** For such ϕ values we have the following picture:

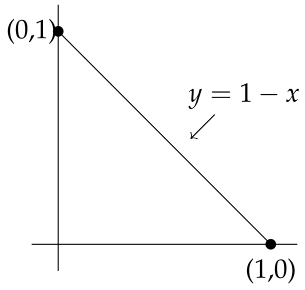
$$z = x^2 + y^2 = r^2 = \rho^2\sin^2\phi \left\{\right.$$



$$
\begin{aligned}
\cos\phi = \frac{z}{\rho} \quad &\Rightarrow \quad \cos\phi = \frac{\rho^2 \sin^2\phi}{\rho} \\
&\Rightarrow \quad \cos\phi = \rho \sin^2\phi \\
&\Rightarrow \quad \frac{\cos\phi}{\sin^2\phi} = \rho
\end{aligned}
$$

$$
\text{Thus Volume} = \int_0^{2\pi}\int_0^{\arctan\frac{1}{3}}\int_0^{9\sec\phi} \rho^2 \sin\phi \; d\rho d\phi d\theta + \int_0^{2\pi}\int_{\arctan\frac{1}{3}}^{\frac{\pi}{2}}\int_0^{\frac{\cos\phi}{\sin^2\phi}} \rho^2 \sin\phi \; d\rho d\phi d\theta
$$

4. A triangular lamina with vertices (0,0), (0,1), and (1,0) has density function $\delta(x, y) = y$. Calculate the mass of the lamina.



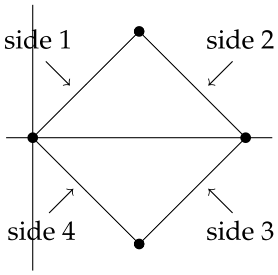
$$
\begin{aligned}
Mass &= \iint\limits_R \delta(x,y)\, dA \\
&= \int_0^1 \int_0^{1-x} y \; dydx \\
&= \int_0^1 \left(\frac{y^2}{2}\right)\Bigg|_0^{1-x} \; dy \\
&= \int_0^1 \left[\frac{(1-x)^2}{2} - 0\right] \; dx \\
&= \frac{1}{2}\int_0^1 \left(1 - 2x + x^2\right) \; dx \\
&= \frac{1}{2}\left(x - x^2 + \frac{x^3}{3}\right)\Bigg|_0^1 \\
&= \frac{1}{2}\left[\left(1 - 1 + \frac{1}{3}\right) - 0\right] \\
&= \frac{1}{6}
\end{aligned}
$$

5. Consider the graph of the function $z = f(x, y) = xy$ defined on the disk of radius $\sqrt{8}$, centered at the origin. Calculate its surface area.

$$
\begin{aligned}
Surface\ area &= \iint\limits_R \sqrt{1+f_x^2+f_y^2}\,dA \\
&= \iint\limits_R \sqrt{1+y^2+x^2}\,dA \\
&\underbrace{=}_{\substack{\text{switch to polar}\\ \text{coordinates}}} \int_0^{2\pi}\int_0^{\sqrt{8}} \underbrace{\sqrt{1+r^2}}_{\sqrt{1+x^2+y^2}}\ \underbrace{(r)\ drd\theta}_{dA} \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\sqrt{8}} 2r\sqrt{1+r^2}\ drd\theta \\
&\underbrace{=}_{\substack{u=1+r^2\\ du=2r\,dr}} \frac{1}{2}\int_0^{2\pi}\int_1^{9} u^{\frac{1}{2}}\ dud\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left.\left(\frac{2}{3}u^{\frac{3}{2}}\right)\right|_1^9\ d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(\frac{2}{3}\right)(27-1)\ d\theta \\
&= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(26)(2\pi) \\
&= \frac{52\pi}{3}
\end{aligned}
$$

6. Let R be the parallelogram in the xy-plane whose vertices are (0,0), (1,1), (2,0), and (1,-1).

   (a) Find a transformation that maps a rectangle, S, in the uv-plane to R in the xy-plane.



Side 1: $y = x \Rightarrow y - x = 0$
Side 2: $y = 2 - x \Rightarrow y + x = 2$

Side 3: $y = x - 2 \Rightarrow y - x = -2$
Side 4: $y = -x \Rightarrow y + x = 0$

On sides 1 and 3, the quantity $y - x$ is constant; let this be u. u ranges from -2 to 0.

On sides 2 and 4 the qunatity y+x is constant; let this be v. v ranges from 0 to 2.

We have u and v in terms of x and y; now get x and y in terms of u and v:

$$\begin{array}{cccc} y & -x & = & u \\ y & +x & = & v \\ \hline 2y & & = & u+v \quad \Rightarrow \quad y = \frac{u+v}{2} \end{array}$$

$$\begin{aligned} y - x = u \quad & \Rightarrow \quad x = y - u \\ & \Rightarrow \quad x = \frac{u+v}{2} - u \\ & \Rightarrow \quad x = \frac{u+v-2u}{2} \\ & \Rightarrow \quad x = \frac{v-u}{2} \end{aligned}$$

Thus we have T(u,v) = $\left(\underbrace{\frac{v-u}{2}}_{x}, \underbrace{\frac{u+v}{2}}_{y} \right)$; $-2 \leq u \leq 0$ and $0 \leq v \leq 2$.

(b) Evaulate $\iint\limits_R (y - x)\ dA_{x,y}$ by using the transformation you found in part (a) to integrate over S.

- $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$
- $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| -\frac{1}{2} \right| = \frac{1}{2}$

$$\begin{aligned} \iint\limits_R (y - x)\ dA_{x,y} &= \iint\limits_S \underbrace{u}_{y-x} \left| \frac{\partial(x,y)}{\partial(u,v)} \right|\ dA_{u,v} \\ &= \int_0^2 \int_{-2}^0 u \left(\frac{1}{2} \right)\ dudv \\ &= \frac{1}{2} \int_0^2 \left. \frac{u^2}{2} \right|_{-2}^{0}\ dv \\ &= \frac{1}{2} \int_0^2 (0 - 2)\ dv \\ &= \left(\frac{1}{2} \right) (-2)(2) \\ &= -2 \end{aligned}$$

7. Let $\mathbf{F} = \langle x, xy, xyz \rangle$

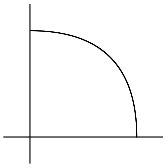
   (a) Calculate div $\mathbf{F}$.

$$\begin{aligned}
div\mathbf{F} &= \bigtriangledown \bullet \mathbf{F} \\
&= \left\langle \tfrac{\partial}{\partial x}, \quad \tfrac{\partial}{\partial y}, \quad \tfrac{\partial}{\partial z} \right\rangle \bullet \langle x, \quad xy, \quad xyz \rangle \\
&= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(xyz) \\
&= 1 + x + xy
\end{aligned}$$

   (b) Calculate curl $\mathbf{F}$.

$$\begin{aligned}
curl\mathbf{F} &= \bigtriangledown \times \mathbf{F} \\
&= \left\langle \tfrac{\partial}{\partial x}, \quad \tfrac{\partial}{\partial y}, \quad \tfrac{\partial}{\partial z} \right\rangle \times \langle x, \quad xy, \quad xyz \rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & xy & xyz \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xyz \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & xyz \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & xy \end{vmatrix} \mathbf{k} \\
&= \left(\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}(xy) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(xyz) \right) \mathbf{j} + \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x) \right) \mathbf{k} \\
&= (xz - 0)\,\mathbf{i} + (0 - yz)\,\mathbf{j} + (y - 0)\,\mathbf{k} \\
&= \langle xz, \quad -yz, \quad y \rangle
\end{aligned}$$

8. Let f(x,y) = xy and let C be the quarter circle of radius 2 shown below, oriented counterclockwise.



   Evaluate $\int_C f(x,y)\ ds$.

   - Parametrize the curve: $\mathbf{r}(t) = \langle 2\cos t, \quad 2\sin t \rangle; 0 \le t \le \frac{\pi}{2}$
   - Calculate $||\mathbf{r}'(t)||$: $\mathbf{r}'(t) = \langle -2\sin t, \quad 2\cos t \rangle$

$$||\mathbf{r}'(t)|| = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4\,\underbrace{\left(\sin^2 t + \cos^2 t\right)}_{1}} = \sqrt{4} = 2$$

   -

$$
\begin{aligned}
\int_C xy\, ds &= \int_0^{\frac{\pi}{2}} \underbrace{(2\cos t)}_{x}\underbrace{(2\sin t)}_{y}\underbrace{(2)}_{||\mathbf{r}'(t)||}\, dt \\
&= 8\int_0^{\frac{\pi}{2}} \cos t \sin t\, dt \\
\underbrace{=}_{\substack{u = \sin t \\ du = \cos t\, dt}} & \; 8\int_0^1 u\, du \\
&= 8\frac{u^2}{2}\Big|_0^1 \\
&= 8\left(\frac{1}{2}\right) \\
&= 4
\end{aligned}
$$