Math 3c, Final Exam

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1. Let $f(x,y) = \sin(xy)$ Find the equation of the plane tangent to the graph of f at the point $\left(\pi, \frac{1}{3}\right)$.

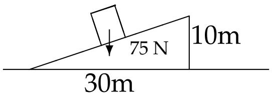
   $f\left(\pi, \frac{1}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

   $f_x(x,y) = y\cos(xy) \quad f_x\left(\pi, \frac{1}{3}\right) = \frac{1}{3}\cos\frac{\pi}{3} = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{6}$

   $f_y(x,y) = x\cos(xy) \quad f_y\left(\pi, \frac{1}{3}\right) = \pi\cos\frac{\pi}{3} = \pi\left(\frac{1}{2}\right) = \frac{\pi}{2}$

   $$\boxed{L(x,y) = \frac{\sqrt{3}}{2} + \frac{1}{6}(x-\pi) + \frac{\pi}{2}\left(y - \frac{1}{3}\right)}$$

2. As shown below, a force of 75 Newtons acts on an object lying on an inclined plane. Calculate the *vector* components of force parallel and perpendicular to the plane. [Answers must be vectors, not numbers!! Do NOT rotate the coordinate axes.]

   F =force vector = $\langle 0, -75\rangle$

   u = unit vector pointing down the plane $= \frac{\langle -3, \quad -1\rangle}{\sqrt{9+1}} = \left\langle -\frac{3}{\sqrt{10}}, \quad -\frac{1}{\sqrt{10}}\right\rangle$
   Note that the components of **u** are both negative, since if we are moving *down* the plane we will be going left and down. The numbers reflect the ratio of sides of the triangle in the picture; we go back 3 for every 1 we go down.

$$\begin{aligned}
\mathbf{p} &= proj_{\mathbf{u}}\mathbf{F} \\
&= (\mathbf{F} \bullet \mathbf{u})\,\mathbf{u} \\
&= \left(\left\langle 0, \quad -75\right\rangle \bullet \left\langle -\tfrac{3}{\sqrt{10}}, \quad -\tfrac{1}{\sqrt{10}}\right\rangle\right) \left\langle -\tfrac{3}{\sqrt{10}}, \quad -\tfrac{1}{\sqrt{10}}\right\rangle \\
&= \frac{75}{\sqrt{10}}\left\langle -\tfrac{3}{\sqrt{10}}, \quad -\tfrac{1}{\sqrt{10}}\right\rangle \\
&= \left\langle -\tfrac{225}{10}, \quad -\tfrac{75}{10}\right\rangle \\
&= \left\langle -\tfrac{45}{2}, \quad -\tfrac{15}{2}\right\rangle
\end{aligned}$$

This is the vector component of force parallel to the plane.

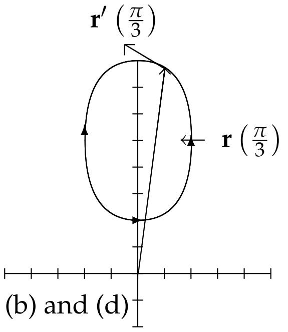
The vector component of force perpendicular to the plane is
$\mathbf{F} - \mathbf{p} = \left\langle 0, \quad -75\right\rangle - \left\langle -\tfrac{45}{2}, \quad -\tfrac{15}{2}\right\rangle = \left\langle \tfrac{45}{2}, \quad -\tfrac{135}{2}\right\rangle$

3. Let $\mathbf{r}(t) = \left\langle 2\cos t, \quad 3\sin t + 5\right\rangle$
   (a) Sketch $\mathbf{r}(t)$, including its orientation.
   This is an ellipse centered at (0,5) with semimajor axis 3 in the vertical direction and semiminor axis 2 in the horizontal direction. It is oriented counterclockwise. See the picture below.
   (b) Sketch $\mathbf{r}\left(\frac{\pi}{3}\right)$ in standard position.
   $\mathbf{r}\left(\frac{\pi}{3}\right) = \left\langle 2\left(\frac{1}{2}\right), \quad 3\left(\frac{\sqrt{3}}{2}\right) + 5\right\rangle = \left\langle 1, \quad \frac{3\sqrt{3}}{2} + 5\right\rangle$
   See graph below.
   (c) Calculate $\mathbf{r}'(t)$.
   $\mathbf{r}'(t) = \left\langle -2\sin t, \quad 3\cos t\right\rangle$
   (d) Calculate $\mathbf{r}'\left(\frac{\pi}{3}\right)$ and sketch this vector in translated position, with its tail at the tip of $\mathbf{r}\left(\frac{\pi}{3}\right)$.
   $\mathbf{r}'\left(\frac{\pi}{3}\right) = \left\langle -2\left(\frac{\sqrt{3}}{2}\right), \quad 3\left(\frac{1}{2}\right)\right\rangle = \left\langle -\sqrt{3}, \quad \frac{3}{2}\right\rangle$
   See graph below.
   (e) Sketch the tangent line to the curve at the point where $t = \frac{\pi}{3}$.
   See graph below.
   (f) Find an equation for this line in vector form.

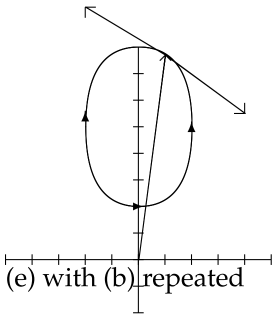
starting point:	$\left\langle 1, \frac{3\sqrt{3}}{2} + 5\right\rangle$
direction vectore:	$\left\langle -\sqrt{3}, \quad \frac{3}{2}\right\rangle$
Line:	$\mathbf{L}(t) = \left\langle 1, \quad \frac{3\sqrt{3}}{2} + 5\right\rangle + t\left\langle -\sqrt{3}, \quad \frac{3}{2}\right\rangle$

   (g) Find an equation for this line by expressing y as a function of x.
   The direction vector: $\left\langle \underbrace{-\sqrt{3}}_{run}, \quad \underbrace{\frac{3}{2}}_{rise}\right\rangle$ has as its components the rise and run of the slope of the tangent line. Thus we have:

$$\begin{array}{ll}\text{point:} & \left(1, \frac{3\sqrt{3}}{2}\right) \\ \text{slope:} & \frac{rise}{run} = \frac{\frac{3}{2}}{-\sqrt{3}} = -\frac{3}{2\sqrt{3}} \\ \text{Line:} & y - \left(\frac{3\sqrt{3}}{2} + 5\right) = -\frac{3}{2\sqrt{3}}(x-1) \\ & y = -\frac{3}{2\sqrt{3}}(x-1) + \frac{3\sqrt{3}}{2} + 5 \end{array}$$



(b) and (d)



(e) with (b) repeated

4. Let G be the portion of the sphere of radius 3 centered at the origin that lies in the first octant. Set up, but don't evaluate, a triple integral to calculate the volume of G:

   For both rectangular and cylindrical coordinates we will use the fact that the projection of G onto the xy-plane is the quarter circle of radius 3 centered at the origin and lying in the first Quadrant.

   (a) in rectangular coordinates.

   $$\int_0^1 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} 1\ dzdydx$$

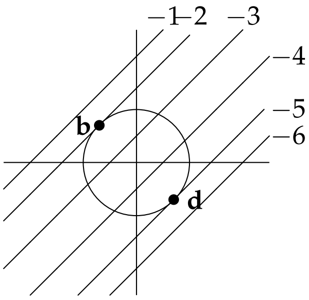
   (b) in cylindrical coordinates.

   $$\int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{\sqrt{9-r^2}} r\ dzdrd\theta$$

   (c) in spherical coordinates.

   $$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^2 \sin\phi\ d\rho d\phi d\theta$$

5. The graph below shows the unit circle together with a contour plot for the function f. Assume that the contour plot gives an accurate representation of f's behavior.



(a) What is the maximum value f achieves on the unit circle?
-2

(b) Put a "b" on the graph at the point where this max is achieved.

(c) What is the minimum value f achieves on the unit circle?
-5

(d) Put a "d" on the graph at the point where this max is achieved.

Explanation: By the Extreme Value Theorem, since the unit circle is closed and bounded, we know that f will achieve a max and min on the unit circle. According to the theory of Lagrange multipliers, any constrained max or min of f will occur at a point where the gradient of f is 0, or where the gradient of f is parallel to the gradient of g. The latter occurs when the level curves share a common tangent line. This occurs at the two points marked on the contour plot. To see which point is the max and which is the min, we look at the values f takes on at these points. **b** is on the level curve $z = -2$ and **d** is on the level curve $z = -5$. Thus -2 is the max and -5 is the min.

6. (a) Given a differentiable function f(P), what vector points in the direction of f's greatest increase?
$\bigtriangledown f$

(b) What is the curvature of a circle of radius 4?
$\frac{1}{4}$ (the reciprocal of the radius)

(c) Given a smooth curve $\mathbf{r}(t); a \leq t \leq b$, what is the formula for calculating the arc length?

$$s = \int_a^b ||\mathbf{r}'(t)||\ dt$$

7. Let $\mathbf{F}(x,y) = \left\langle \underbrace{3x^2y^2 - \sin x,}_{f} \ \underbrace{2x^3y - e^y}_{g} \right\rangle$.

(a) Verify that **F** is conservative.
$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(3x^2y^2 - \sin x\right) = 6x^2y$

$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(2x^3y - e^y\right) = 6x^2y$
$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ so **F** is conservative.

(b) Find a potential function for **F**.

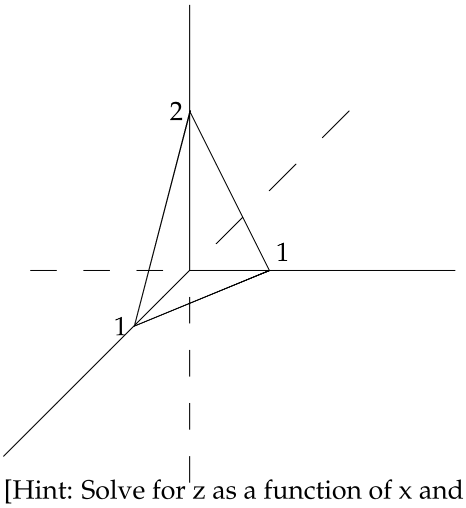
Since **F** is conservative, we know there exists a potential function ϕ such that
$\frac{\partial \phi}{\partial x} = f$ and $\frac{\partial \phi}{\partial y} = g$. Thus we can find ϕ by integrating either function with respect to the appropriate variable.

$$\begin{aligned}\phi &= \int \frac{\partial \phi}{\partial x}\, dx \\ &= \int \left(3x^2y^2 - \sin x\right)\, dx \\ &= x^3y^2 + \cos x + k(y)\end{aligned}$$

Since we were integrating with respect to x, any function of y is considered constant, so instead of adding "+c" to a particular antiderivative, we add an arbitrary function of y. Now we need to determine what this function is.
We know that $\frac{\partial \phi}{\partial y} = g = 2x^3y - e^y$
We also know that $\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(x^3y^2 + \cos x + k(y)\right) = 2x^3y + k'(y)$
Setting the two expressions for $\frac{\partial \phi}{\partial y}$ equal to each other we get
$2x^3y - e^y = 2x^3y + k'(y) \Rightarrow k'(y) = -e^y$
Thus $k(y) = \int -e^y\, dy = -e^y + c$
Since we only need to find *a* potential function, we can choose to set c=0.
$\phi = x^3y^2 + \cos x - e^y$.

8. Let σ be the portion of the plane $2x + 2y + z = 2$ lying above the first octant. Evaluate

$$\iint_{\sigma} x + z\, dS$$

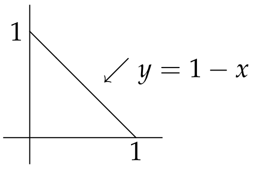


[Hint: Solve for z as a function of x and y.]

$z = 2 - 2x - 2y = f(x, y)$

$f_x = -2 \; and \; f_y = -2$

Let R be the projection of σ onto the xy-plane. Then R is the triangle shown below.



$$
\begin{aligned}
\iint_{\sigma} x + z\, dS &= \iint_{R} (x + (2 - 2x - 2y))\sqrt{(-2)^2 + (-2)^2 + 1}\, dA \\
&= \int_0^1 \int_0^{1-x} (2 - x - 2y)\sqrt{5}\, dy dx \\
&= \sqrt{5}\int_0^1 \left(2y - xy - y^2\right)\Big|_0^{1-x}\, dx \\
&= \sqrt{5}\int_0^1 \left(2(1-x) - x(1-x) - \left(1 - 2x + x^2\right)\right)\, dx \\
&= \sqrt{5}\int_0^1 \left(2 - 2x - x + x^2 - 1 + 2x - x^2\right)\, dx \\
&= \sqrt{5}\int_0^1 (1 - x)\, dx \\
&= \sqrt{5}\left(x - \frac{x^2}{2}\right)\Big|_0^1 \\
&= \sqrt{5}\left(1 - \frac{1}{2}\right) \\
&= \frac{\sqrt{5}}{2}
\end{aligned}
$$

9. (a) Let $\mathbf{F}(x, y, z)$ be a vector field whose component functions have continuous first partials. Let G be a finite solid whose boundary surface σ is piecewise smooth and oriented outward. State the Divergence Theorem.

$$\iint_{\sigma} \mathbf{F} \bullet \mathbf{n}\, dS = \iiint_{G} div\mathbf{F}\, dV$$

(b) Use the Divergence Theorem to evaluate the flux of $\mathbf{F}(x, y, z) = \langle xz^2, \ yz^2, \ z^3 \rangle$ across the outward-oriented sphere of radius 2, centered at the origin.

$div\mathbf{F} = \bigtriangledown \bullet \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ \frac{\partial}{\partial z} \right\rangle \bullet \langle xz^2, \ yz^2, \ z^3 \rangle = \frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(z^3) = z^2 + z^2 + 3z^2 = 5z^2$

$$\begin{aligned}
\iint\limits_{\sigma} \mathbf{F} \bullet \mathbf{n}\, dS &\underbrace{=}_{\substack{\text{Divergence}\\ \text{Theorem}}} \iiint\limits_{G} div\mathbf{F}\, dV \\
&= \iiint\limits_{G} 5z^2\, dV \\
&\underbrace{=}_{\substack{\text{switch to}\\ \text{spherical}\\ \text{coordinates}}} \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho^2 \sin\phi \underbrace{\left(5\rho^2\cos^2\phi\right)}_{5z^2} d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} 5\rho^4 \sin\phi \cos^2\phi\, d\rho d\phi d\theta \\
&= 5\int_0^{2\pi}\int_0^{\pi} \sin\phi\cos^2\phi \left.\frac{\rho^5}{5}\right|_0^2 d\phi d\theta \\
&= 5\int_0^{2\pi}\int_0^{\pi} \sin\phi\cos^2\phi \left(\frac{32}{5}\right) d\phi d\theta \\
&\underbrace{=}_{\substack{u=\cos\phi\\ u'=-\sin\phi\\ du=-\sin\phi\, d\phi}} -32\int_0^{2\pi}\int_1^{-1} u^2\, du d\theta \\
&= -32\int_0^{2\pi} \left.\frac{u^3}{3}\right|_1^{-1} d\theta \\
&= -32\int_0^{2\pi} \left(\frac{-1}{3}-\frac{1}{3}\right) d\theta \\
&= (32)\left(\frac{2}{3}\right)\int_0^{2\pi} 1\, d\theta \\
&= \frac{64}{3}(2\pi) \\
&= \frac{128\pi}{3}
\end{aligned}$$

10. Let σ be the top half of the unit sphere, centered at the origin, with outward orientation, with boundary curve C, positively oriented. Let $\mathbf{F}(x,y,z) = \langle x+y, y+z, x+z \rangle$. Calculate $\int_C \mathbf{F} \bullet \mathbf{T}\, ds$ two ways:

    (a) By evaluating this as a line integral.

    C is the curve where the unit sphere intersects the xy-plane; this is the unit circle.

Since C is oriented outward, the unit circle is oriented counterclockwise by the right-hand rule.

Thus, we can parametrize C as follows: $\mathbf{r}(t) = \langle \cos t, \quad \sin t, \quad 0\rangle$
$0 \le t \le 2\pi$

Thus $\mathbf{r}'(t) = \langle -\sin t, \quad \cos t\rangle$.

$$
\begin{aligned}
\int_C \mathbf{F} \bullet \mathbf{T}\, ds \quad &= \quad \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\
&= \quad \int_0^{2\pi} \left\langle \underbrace{\cos t + \sin t}_{x+y}, \quad \underbrace{\sin t}_{y+z}, \quad \underbrace{\cos t}_{x+y} \right\rangle \bullet \langle -\sin t, \quad \cos t, \quad 0\rangle\; dt \\
&= \quad \int_0^{2\pi} \Big(-\cos t\; \sin t - \sin^2 t + \sin t\; cost \Big)\; dt \\
&= \quad \int_0^{2\pi} -\sin^2 t\; dt \\
&\underbrace{=}_{\text{double angle formula}} \quad -\int_0^{2\pi} \frac{1-\cos 2t}{2}\, dt \\
&= \quad -\frac{1}{2}\int_0^{2\pi} (1-\cos 2t)\; dt \\
&\underbrace{=}_{\substack{u\,=\,2t \\ du\,=\,2\,dt}} \quad -\frac{1}{2}\left(\frac{1}{2}\right)\int_0^{4\pi} (u - \cos u)\; du \\
&= \quad -\frac{1}{4}\,(u - \sin u)\Big|_0^{4\pi} \\
&= \quad -\frac{1}{4}\,[(4\pi - 0) - 0] \\
&= \quad -\pi
\end{aligned}
$$

(b) By using Stokes' theorem to evaluate an appropriate surface integral.

According to Stokes' theorem, $\int_C \mathbf{F} \bullet \mathbf{T}\, ds = \iint\limits_{\sigma} curl\mathbf{F} \bullet \mathbf{n}\, dS$

We *could* work with the given surface, but strategic laziness and Stokes' theorem suggest that we should pick the easiest surface we can think of whose positively oriented boundary is C.

Let $\widetilde{\sigma}$ be the unit disk in the xy-plane, viewed as a surface in 3-space. Since C is oriented counterclockwise, we use the right-hand rule to determine that $\widetilde{\sigma}$ must be oriented up. Since $\widetilde{\sigma}$ lies in the xy-plane, this means that $\mathbf{n}$ = its principal unit normal vector = $\mathbf{k} = \langle 0, \quad 0, \quad 1\rangle$

$$curl\mathbf{F} = \bigtriangledown \times \mathbf{F} =$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & y+z & x+z \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & x+z \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x+y & x+z \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x+y & y+z \end{vmatrix} \mathbf{k}$$

$$= -1\mathbf{i} - \mathbf{j} - \mathbf{k} = \left\langle -1, \quad -1, \quad -1 \right\rangle$$

$$\begin{aligned} \int_C \mathbf{F} \bullet \mathbf{T}\, ds \quad & \underbrace{=}_{\text{Stokes' Theorem}} \quad \iint\limits_{\widetilde{\sigma}} curl\mathbf{F} \bullet \mathbf{n}\, dS \\ & = \quad \iint\limits_{\widetilde{\sigma}} \left\langle -1, \quad -1, \quad -1 \right\rangle \bullet \left\langle 0, \quad 0, \quad 1 \right\rangle\, dS \\ & = \quad \iint\limits_{\widetilde{\sigma}} -1\, dS \\ & = \quad -\left(area\ of\ \ \widetilde{\sigma} \right) \\ & = \quad -1\left(\pi\right) \\ & = \quad -\pi \end{aligned}$$