Solutions: Math 3c, Spring 2007 Final Exam, Karla Westphal

1. Let $\mathbf{r}(t) = \langle e^t, \quad t^2 + t \rangle$.

   (a) Find a formula for $\frac{dy}{dx}$.

   - $\frac{dy}{dt} = 2t + 1$
   - $\frac{dx}{dt} = e^t$
   - $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{e^t}$

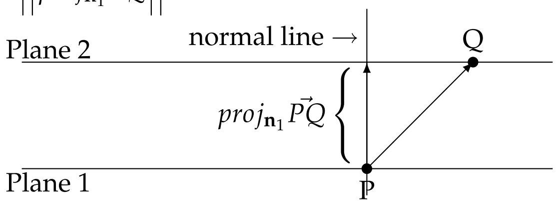
   (b) Find a formula for $\frac{d^2y}{dx^2}$.

   - Let $y' = \frac{dy}{dx} = \frac{2t+1}{e^t}$
   - $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$
   - $\frac{dy'}{dt} \underbrace{=}_{\text{quotient rule}} \frac{2e^t-(2t+1)e^t}{e^{2t}}$
   - $\frac{dx}{dt} \underbrace{=}_{\text{same as above}} e^t$
   - $\frac{d^2y}{dx^2} = \frac{\frac{2e^t-2te^t-e^t}{e^{2t}}}{e^t} = \frac{e^t-2te^t}{e^{3t}}$

2. Calculate the distance between the parallel planes given below:

$$\begin{array}{lll} \textit{Plane } 1 & 2x - y + 3z = 4 & \mathbf{n}_1 = \langle 2, \quad -1, \quad 3 \rangle \\ \textit{Plane } 2: & -2x + y - 3z = 5 & \mathbf{n}_2 = \langle -2, \quad 1, \quad -3 \rangle \end{array}$$

- Note that the two planes share a normal direction: $\mathbf{n}_2 = -\mathbf{n}_1$. (Of course, if they didn't, they wouldn't be parallel.)
- Pick a point P on Plane 1 and a Point Q on Plane 2. (Choices of points may vary.)
  P: (2,0,0)
  Q: (0,5,0)
  $\vec{PQ} = \langle -2, \quad 5, \quad 0 \rangle$
- Translate $\mathbf{n}_1$ (or equivalently $\mathbf{n}_2$) so that its tail is at P. In the picture below, the line determined by this translated vector is drawn in and labelled the normal line.
  distance = $\left|\left| proj_{\mathbf{n}_1} \vec{PQ} \right|\right|$



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$$\begin{aligned} proj_{\mathbf{n}_1}\vec{PQ} &= \left(\frac{\vec{PQ} \bullet \mathbf{n}_1}{\mathbf{n}_1 \bullet \mathbf{n}_1}\right)\mathbf{n}_1 \\ &= \left(\frac{\langle 2, \quad -1, \quad 3\rangle \bullet \langle -2, \quad 5, \quad 0\rangle}{\langle 2, \quad -1, \quad 3\rangle \bullet \langle 2, \quad -1, \quad 3\rangle}\right)\langle 2, \quad -1, \quad 3\rangle \\ &= \left(\frac{-9}{14}\right)\langle 2, \quad -1, \quad 3\rangle \end{aligned}$$

- $\left|\left|proj_{\mathbf{n}_1}\vec{PQ}\right|\right| = |-\frac{9}{14}|\sqrt{14} = \boxed{\frac{9\sqrt{14}}{14}}$

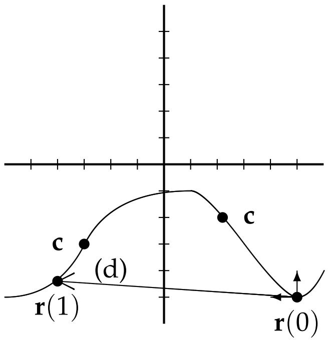
3. Let $\mathbf{r}(t) = \left\langle \frac{t^3}{3}, \quad \frac{t^2}{2}\right\rangle; t \geq 0$. Find the arc length parametrization for this curve with t=0 as the reference point.

$$\begin{aligned} \mathbf{r}(u) &= \left\langle \frac{u^3}{3}, \quad \frac{u^2}{2}\right\rangle; u \geq 0 \\ \mathbf{r}'(u) &= \langle u^2, \quad u\rangle \\ ||\mathbf{r}'(u)|| &= \sqrt{(u^2)^2 + u^2} \\ &= \sqrt{u^2(u^2+1)} \\ &= |u|\sqrt{u^2+1} \\ &\underbrace{=}_{u \geq 0} u\sqrt{u^2+1} \end{aligned}$$

$$\begin{aligned}
s &= \int_0^t ||\mathbf{r}'(u)||\ du \\
&= \int_0^t u\sqrt{u^2+1}\,du \\
&\underbrace{=}_{\substack{w=u^2+1\\ dw=2udu}} \frac{1}{2}\int_1^{t^2+1} w^{\frac{1}{2}}\ dw \\
&= \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_1^{t^2+1} \\
&= \frac{1}{3}\left(\left(t^2+1\right)^{\frac{3}{2}}-1\right) \\
3s &= \left(t^2+1\right)^{\frac{3}{2}}-1 \\
3s+1 &= \left(t^2+1\right)^{\frac{3}{2}} \\
(3x+1)^{\frac{2}{3}} &= t^2+1 \\
(3x+1)^{\frac{2}{3}}-1 &= t^2 \\
\sqrt{(3x+1)^{\frac{2}{3}}-1} &\underbrace{=}_{\substack{t\geq 0 \text{ so}\\ \sqrt{t^2}=|t|=t}} t
\end{aligned}$$

$$\boxed{\mathbf{r}(s) = \left\langle \frac{\left(\sqrt{(3x+1)^{\frac{2}{3}}-1}\right)^3}{3},\ \frac{(3x+1)^{\frac{2}{3}}-1}{2} \right\rangle}$$

4. The graph of a smooth curve $\mathbf{r}(t)$ is drawn below, with two points (drawn as points rather than vectors) indicated. For this problem, you may record your answers directly on the test. Assume that each tick mark on the grid indicates a lenght of one unit.



(a) Draw $\mathbf{T}(0)$ in translated position so that its tail is at the tip of $\mathbf{r}(0)$.

$\mathbf{T}(0)$ is the vector pointing to the left (tangent to the curve, in the direction of increasing parameter.)

(b) Draw $\mathbf{N}(0)$ in translated position so that its tail is at the tip of $\mathbf{r}(0)$.

$\mathbf{N}(0)$ is the vector pointing up (perpendicular to $\mathbf{T}(0)$ and pointing to the concave side of the curve.

(c) Put a "**c**" on the graph at a point where the unit normal vector does not exist.

Two possible locations are shown; since $\mathbf{N}(t)$ must point to the concave side of the curve, $\mathbf{N}(t)$ will not exist at points where the concavity changes.

(d) Draw the vector $\int_0^1 \mathbf{r}'(t)\,dt$ in translated position so that its tail is at the tip of $\mathbf{r}(0)$.

This is the vector that points from $\mathbf{r}(0)$ to $\mathbf{r}(1)$.

(e) Which is bigger: $\kappa(0)$ or $\kappa(1)$?

$\kappa(0)$ is bigger; the radius of curvature is smaller, and curvature is the reciprocal of the radius of curvature.

5. Let R be the region enclosed by the triangle whose vertices are (-1,0), (1,0), and (-1,4). Let $f(x,y)$ = x-xy.

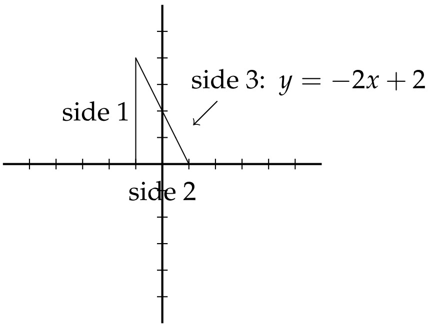
(a) How do we know that f achieves an absolute max and an absolute min on R? [Be specific; name the theorem and tell me what conditions must be satisfied for the theorem to apply.]

The Extreme Value Theorem ensures the existence of an absolute max and min since f is continuous and R is closed and bounded.

(b) Find the location and value of the absolute max and the absolute min of f on R.

- The max and min can occur at two types of locations: critical points and boundary points.

- Critical points:
  $\bigtriangledown f = \langle f_x, \quad f_y \rangle = \langle 1-y, \quad -x \rangle$
  $\bigtriangledown f$ is always defined.
  $\bigtriangledown f = \mathbf{0} \Rightarrow 1-y = 0 \textit{ and } -x = 0 \Rightarrow y = 1 \textit{ and } x = 0$
  (0,1) is a critical point. As the picture below shows, this point *is* contained in R.
- Boundary: The boundary consists of 3 sides, labelled in the picture below.
  Along each side of the boundary we will need to consider critical points and boundary points.



Side 1:

$$\begin{aligned} \mathbf{r}(t) &= \langle -1 \quad t \rangle ; 0 \le t \le 4 \\ f(\mathbf{r}(t)) &= -1 + t \\ f'(t) &= 1 \end{aligned}$$

Along side 1 we have no critical points, but we must check the endpoints: t = 0 and t = 4.

Side 2:

$$\begin{aligned} \mathbf{r}(t) &= \langle t, \quad 0 \rangle ; -1 \le t \le 1 \\ f(\mathbf{r}(t)) &= t - 0 \\ &= t \\ f'(t) &= 1 \end{aligned}$$

$f'(t) \neq 0$
Thus we need only consider the endpoints of this segment; there are no critical points on side 2.

Side 3:

$$\begin{aligned}
\mathbf{r}(t) &= \langle t,\quad -2t+2\rangle\,; -1 \le t \le 1 \\
f\left(\mathbf{r}(t)\right) &= t - t(-2t+2) \\
&= t + 2t^2 - 2t \\
&= 2t^2 - t \\
f'(t) &= 4t - 1
\end{aligned}$$

$f'(t)$ is always defined; $f'(t) = 0 \Rightarrow t = \frac{1}{4}$

- Putting it all together:

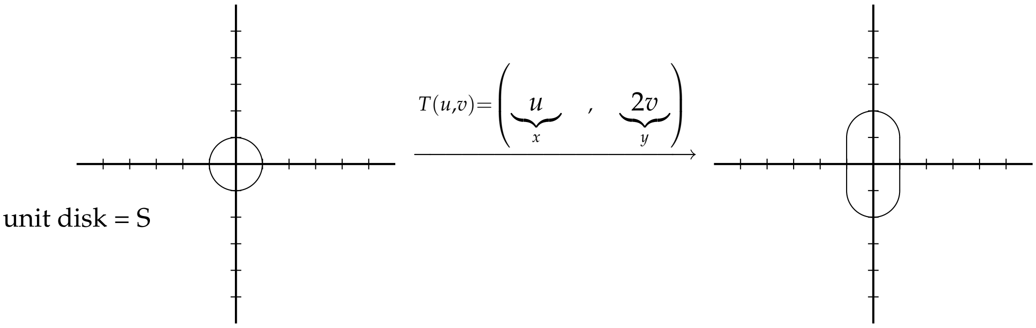
		t (boundary points only)	(x,y)	f(x,y)
critical point			(0,1)	0
boundary points	side 1	0 (endpoint)	(-1,0)	-1
		4(endpoint)	(-1,4)	3
	side 2	-1 (endpoint)	$(-1,0)$	0
		1 (endpoint)	(1,0)	1
	side 3	$\frac{1}{4}$ (critical point)	$\left(\frac{1}{4},\frac{3}{2}\right)$	$-\frac{1}{8}$
		-1 (endpoint)	(-1,4)	0
		1(endpoint)	(1,0)	1

- From the chart above, we see that the minimum value f achieves on R is -1; this point is achieved at $(-1,0)$.
  The maximum value f achieves on R is 3; this value is achieved at $(-1,4)$.

6. Let R be the region enclosed by the ellipse $x^2 + \frac{y^2}{4} = 1$. Evaluate

$$\iint\limits_R (xy+2)\,dA$$

[Hint: Start by finding a transformation that maps the unit disk to R.]



$x = u$
$y = 2v$

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \\
&= 2 \\
\left|\frac{\partial(x,y)}{\partial(u,v)}\right| &= |2| \\
&= 2
\end{aligned}$$

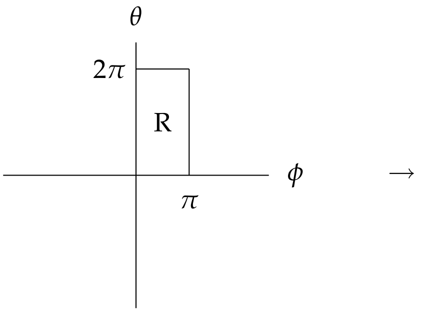
$$\begin{aligned}
\iint\limits_R (xy+2)\,dA_{x,y} &= \iint\limits_S \left(\underbrace{(u)}_{x}\underbrace{(2v)}_{y}+2\right)\underbrace{2}_{\left|\frac{\partial(x,y}{\partial(u,v)}\right|}\,dA_{u,v} \\
&\underbrace{=}_{\text{switch to polar}} 2\int_0^{2\pi}\int_0^1 \left(\underbrace{(r\cos\theta)}_{u}\underbrace{(2r\sin\theta)}_{2v}+2\right)(r)\,drd\theta \\
&= 2\int_0^{2\pi}\int_0^1 \left(2r^3\cos\theta\sin\theta+2r\right)\,drd\theta \\
&= 2\int_0^{2\pi}\int_0^1 \left(\frac{r^4}{2}\cos\theta\sin\theta+r^2\right)\Bigg|_0^1\,d\theta \\
&= 2\int_0^{2\pi}\left(\frac{1}{2}\cos\theta\sin\theta+1\right)\,d\theta \\
&= 2\int_0^{2\pi}\frac{1}{2}\cos\theta\sin\theta\,d\theta+2\int_0^{2\pi}2\,d\theta \\
&\underbrace{=}_{\substack{u=\sin\theta \\ du=\cos\theta d\theta}} 2\int_0^0 u\,du+2(2)(area\ of\ S) \\
&= 0+4(\pi) \\
&= 4\pi
\end{aligned}$$

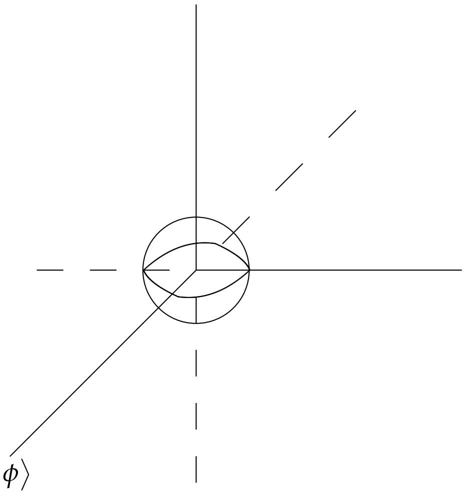
7. Let σ be the surface of the unit sphere, centered at the origin. Evaluate:

$$\iint\limits_{\sigma} z^2\,dS$$

First, parametrize the unit sphere in terms of ϕ and θ.

$$\mathbf{r}(\phi,\theta) = \left\langle \sin\phi\cos\theta, \quad \sin\phi\sin\theta, \quad \cos\phi \right\rangle; \quad \begin{array}{l} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{array}$$





$$\frac{\partial \mathbf{r}}{\partial \phi} = \left\langle \cos\phi\cos\theta, \quad \cos\phi\sin\theta \quad -\sin\phi \right\rangle$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = \left\langle -\sin\phi\sin\theta, \quad \sin\phi\cos\theta, \quad 0 \right\rangle$$

$$\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} =$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} = \begin{vmatrix} \cos\phi\sin\theta & -\sin\phi \\ \sin\phi\cos\theta & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \cos\phi\cos\theta & -\sin\phi \\ -\sin\phi\sin\theta & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta \\ -\sin\phi\sin\theta & \sin\phi\cos\theta \end{vmatrix} \mathbf{k}$$

$$= \left\langle \sin^2\phi\cos\theta, \quad \sin^2\phi\sin\theta, \quad \underbrace{\cos\phi\sin\phi\cos^2\theta + \cos\phi\sin\phi\sin^2\theta}_{\cos\phi\sin\phi\left(\cos^2\theta+\sin^2\theta\right)} \right\rangle$$

$$= \left\langle \sin^2\phi\cos\theta, \quad \sin^2\phi\sin\theta, \quad \cos\phi\sin\phi \right\rangle$$

$$\begin{aligned}
\left|\left|\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}\right|\right| &= \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \cos^2\phi\sin^2\phi} \\
&= \sqrt{\sin^4\phi\underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{1} + \cos^2\phi\sin^2\phi} \\
&= \sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi} \\
&= \sqrt{\sin^2\phi\underbrace{\left(\sin^2\phi + \cos^2\phi\right)}_{1}} \\
&= \sqrt{\sin^2\phi} \\
&\underbrace{=}_{\substack{\sin\phi \geq 0 \\ if \\ 0 \leq \phi \leq \pi}} \sin\phi
\end{aligned}$$

$$\begin{aligned}
\iint\limits_{\sigma} z^2\, dS &= \iint\limits_{R} \left(\underbrace{\cos^2\phi}_{z^2}\right)\sin\phi\, dA \\
&= \int_0^{2\pi}\int_0^{\pi} \cos^2\phi\sin\phi\, d\phi d\theta \\
&\underbrace{=}_{\substack{u = \cos\phi \\ du = -\sin\phi\, d\phi}} -\int_0^{2\pi}\int_1^{-1} u^2\, du d\theta \\
&= -\int_0^{2\pi}\left(\frac{u^3}{3}\right)\Bigg|_1^{-1} d\theta \\
&= -\int_0^{2\pi}\left(-\frac{2}{3}\right) d\theta \\
&= -\left(-\frac{2}{3}\right)(2\pi) \\
&= \frac{4\pi}{3}
\end{aligned}$$

8. Let $\mathbf{F}(x,y) = \left\langle \underbrace{ye^{xy} + \sec^2 x}_{f}, \quad \underbrace{xe^{xy} - 2}_{g} \right\rangle$.

(a) Verify that **F** is conservative.

- $f_y = e^{xy} + xye^{xy}$
- $g_x = e^{xy} + xye^{xy}$
- $f_y = g_x$ Thus **F** is conservative.

That is, $\mathbf{F} = \triangledown\phi = \left\langle \underbrace{ye^{xy} + \sec^2 x,}_{\frac{\partial\phi}{\partial x}} \quad \underbrace{xe^{xy} - 2}_{\frac{\partial\phi}{\partial y}} \right\rangle$

(b) Find a potential function ϕ for **F**.

Integrate.

$$\begin{aligned}
\phi &= \int \frac{\partial\phi}{\partial x}\,dx \\
&= \int \left(ye^{xy} + \sec^2 x\right)\,dx \\
&= \int ye^{xy}\,dx + \int \sec^2 x\,dx \\
&\underbrace{=}_{\substack{u = xy \\ du = y\,dx}} \int e^u\,du + \tan x + \underbrace{k(y)}_{\text{constant with respect to x}} \\
&= e^u + \tan x + k(y) \\
&= e^{xy} + \tan x + k(y)
\end{aligned}$$

Differentiate.

$$\begin{aligned}
\frac{\partial\phi}{\partial y} &= \frac{\partial}{\partial y}\left(e^{xy} + \tan x + k(y)\right) \\
&= xe^{xy} + k'(y)
\end{aligned}$$

Setting this equal to g, which we know is equal to $\frac{\partial\phi}{\partial y}$, we get:

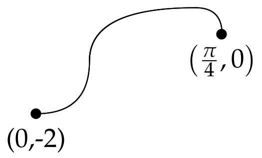
$$xe^{xy} + k'(y) = xe^{xy} - 2 \Rightarrow \quad k'(y) = -2$$

Integrate.

$$\begin{aligned}
k(y) &= \int k'(y)\,dy \\
&= \int -2\,dy \\
&= -2y + c
\end{aligned}$$

Thus an antiderivative (choosing c to be 0) is given by
$\phi = e^{xy} + \tan x - 2y$

(c) Use your answer to part (b) to evaluate $\int_C \mathbf{F} \bullet \mathbf{T}\, ds$ where C is the curve shown below traced from left to right:



$$\begin{aligned}\int_C \mathbf{F} \bullet \mathbf{T}\, ds &= \phi\left(\frac{\pi}{4}, 0\right) - \phi(0,-2) \\ &= (1+1-0) - (1+0+4) \\ &= 2-5 \\ &= -3\end{aligned}$$

9. Let G be the unit cube located in the first octant with one corner at the origin. Let σ be the boundary surface of this cube and let σ have outward orientation. Let $\mathbf{E} = \langle xy, \quad z-y, \quad x^2y^3 \rangle$. Calculate the flux of $\mathbf{E}$ across σ using any (legitimate) method.

$$\begin{aligned} div\mathbf{E} &= \bigtriangledown \bullet \mathbf{E} \\ &= \left\langle \tfrac{\partial}{\partial x}, \quad \tfrac{\partial}{\partial y}, \quad \tfrac{\partial}{\partial z} \right\rangle \bullet \langle xy, \quad z-y, \quad x^2y^3 \rangle \\ &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(z-y) + \frac{\partial}{\partial z}\left(x^2y^3\right) \\ &= y-1 \end{aligned}$$

$$
\begin{aligned}
Flux = \Phi &= \iint_{\sigma} \mathbf{E} \bullet \mathbf{n}\, dS \\
&\underbrace{=}_{\substack{\text{Divergence}\\ \text{Theorem}}} \iiint_{G} div\mathbf{E}\, dV \\
&= \int_0^1 \int_0^1 \int_0^1 (y-1)\; dydxdz \\
&= \int_0^1 \int_0^1 \left(\frac{y^2}{2} - y\right)\Bigg|_0^1 \; dxdz \\
&= \int_0^1 \int_0^1 \left(\frac{1}{2} - 1\right)\; dxdz \\
&= \left(-\frac{1}{2}\right) \text{(area of unit square in xz-plane)} \\
&= \left(-\frac{1}{2}\right)(1) \\
&= -\frac{1}{2}
\end{aligned}
$$

10. Let σ be portion of the paraboloid $z = x^2 + y^2 - 9$ lying on and below the xy-plane. Let σ have downward orientation. Let C be its boundary curve, oriented positively with respect to σ. Let $\mathbf{F} = \langle x - y, \quad x^2z, \quad y \rangle$.

    (a) Evaluate $\int_C \mathbf{F} \bullet \mathbf{T}\, ds$ as a line integral.
    C = boundary curve is the circle where the paraboloid meets the xy-plane:
    $0 = x^2 + y^2 - 9 \Rightarrow x^2 + y^2 = 9$
    This is the circle of radius 3, centered at the origin.
    Since σ has *downward* orientation, C must be oriented *clockwise*.
    Thus we can parametrize C as follows: $\mathbf{r}(t) = \langle 3\cos t, \quad -3\sin t, \quad 0 \rangle\,; 0 \le t \le 2\pi$
    $\mathbf{r}'(t) = \langle -3\sin t, \quad -3\cos t, \quad 0 \rangle$

$\int_C \mathbf{F} \bullet \mathbf{T}\, ds$

$$= \int_0^{2\pi} \mathbf{F} \bullet \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \left\langle \underbrace{3\cos t - (-3\sin t)}_{x-y},\quad \underbrace{(3\cos t)^2(0)}_{x^2z},\quad \underbrace{-3\sin t}_{y} \right\rangle \bullet \left\langle -3\sin t, \quad -3\cos t, \quad 0 \right\rangle\, dt$$

$$= \int_0^{2\pi} \left\langle 3\cos t + 3\sin t, \quad 0, \quad -3\sin t \right\rangle \bullet \left\langle -3\sin t, \quad -3\cos t, \quad 0 \right\rangle\, dt$$

$$= \int_0^{2\pi} \left(-9\cos t \sin t + -9\sin^2 t \right)\, dt$$

$$= -9\int_0^{2\pi} \cos t \sin t\, dt - 9\int_0^{2\pi} \sin^2 t\, dt$$

$$\underbrace{=}_{\substack{u = \sin t \\ du = \cos t\, dt}} \quad -9\int_0^0 u\, du - 9\int_0^{2\pi} \underbrace{\left(\frac{1-\cos 2t}{2} \right)}_{\text{double angle formula}}\, dt$$

$$\underbrace{=}_{\substack{w = 2t \\ dw = 2\, dt}} \quad 0 - 9\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\int_0^{4\pi} (1-\cos w)\, dw$$

$$= \left(-\tfrac{9}{4}\right)\, (w - \sin w)\big|_0^{4\pi}$$

$$= \left(-\tfrac{9}{4}\right)(4\pi)$$

$$= -9\pi$$

(b) Use Stokes' theorem to evaluate $\int_C \mathbf{F} \bullet \mathbf{T}\, ds$ by evaluating an appropriate surface integral.

We *could* use the original surface, σ, but it'll be much easier to use a simpler surface with the same boundary curve.

A simpler surface with the same boundary is the disk of radius 3, centered at the origin in the xy-plane. To preserve the orientation of C, we will have to orient this surface *downward* so that the unit normal vector $\mathbf{n}$ is $-\mathbf{k} = \langle 0, \quad 0, \quad -1\rangle$. Let's call this oriented surface $\tilde{\sigma}$; its projection onto the xy-plane (essentially itself, but viewed as living in $\mathbb{R}^2$rather than $\mathbb{R}^3$) we will call R.

$$curl\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & x^2z & y \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & y \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x-y & y \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x-y & x^2z \end{vmatrix} \mathbf{k}$$

$$= \underbrace{\left\langle *, \quad **, \quad \tfrac{\partial}{\partial x}\left(x^2z\right) - \tfrac{\partial}{\partial y}\left(x-y\right) \right\rangle}_{\substack{\text{the first two components are unimportant, as} \\ \text{they will cancel out when we dot with -}\mathbf{k}}} = \langle *, \quad **, \quad 2xz+1 \rangle$$

$$
\begin{aligned}
\int_C \mathbf{F} \bullet \mathbf{T}\, ds \quad & \underbrace{=}_{\text{Stokes' theorem}} \quad \iint_{\tilde{\sigma}} curl\mathbf{F} \bullet \mathbf{n}\, dS \\
& = \quad \iint_{\tilde{\sigma}} \langle *, \quad **, \quad 2xz+1\rangle \bullet \langle 0, \quad 0, \quad -1\rangle \; dS \\
& = \quad \iint_{\tilde{\sigma}} (-2xz-1)\; dS \\
& \underbrace{=}_{\substack{\text{the scaling factor}\\ \text{as we switch}\\ \text{from dS to dA is}\\ \text{just 1 in this case}}} \quad \iint_R \quad \underbrace{-1}_{\substack{\text{the term 2xz be-}\\ \text{comes 0 since}\\ \text{z=0 on this sur-}\\ \text{face}}} \quad dA \\
& = \quad -area(A) \\
& = \quad -\pi(3)^2 \\
& = \quad -9\pi
\end{aligned}
$$