Section 13.2

(54) Prove $\frac{d}{dt}[\vec{u}\cdot(\vec{v}\times\vec{w})] = \vec{u}'\cdot(\vec{v}\times\vec{w}) + \vec{u}\cdot(\vec{v}'\times\vec{w}) + \vec{u}\cdot(\vec{v}\times\vec{w}')$

Proof:

$$\frac{d}{dt}[\vec{u}\cdot(\vec{v}\times\vec{w})] = \vec{u}'\cdot(\vec{v}\times\vec{w}) + \vec{u}\cdot\frac{d}{dt}[\vec{v}\times\vec{w}]$$

by Formula 6

$$= \vec{u}'\cdot(\vec{v}\times\vec{w}) + \vec{u}\cdot[\vec{v}'\times\vec{w} + \vec{v}\times\vec{w}']$$

by Formula 7

$$= \vec{u}'\cdot(\vec{v}\times\vec{w}) + \vec{u}\cdot(\vec{v}'\times\vec{w}) + \vec{u}\cdot(\vec{v}\times\vec{w}')$$

property of dot product

as needed.

(56) Prove Thm 13.2.6 in $\mathbb{R}^2$.

Let $\vec{r}(t)$, $\vec{r}_1(t)$, $\vec{r}_2(t)$ be vector-valued fn. in $\mathbb{R}^2$.
Let $f(t)$ be a real-valued fn., k a scalar & $\vec{c}$ a constant vector.

(a) $\frac{d}{dt}[\vec{c}] = \frac{d}{dt}\langle c_1, c_2\rangle$

$= \langle \frac{d}{dt}[c_1], \frac{d}{dt}[c_2]\rangle$

$= \langle 0, 0\rangle$

$= \vec{0}$ as needed.

(b) Done in class lecture notes.

$\vec{r}_1(t) = \langle x_1(t), y_1(t)\rangle$ & $\vec{r}_2(t) = \langle x_2(t), y_2(t)\rangle$

(c) & (d) $\frac{d}{dt}[\vec{r}_1(t) \pm \vec{r}_2(t)] = \frac{d}{dt}[\langle x_1(t) \pm x_2(t), y_1(t) \pm y_2(t)\rangle]$

$= \langle \frac{d}{dt}[x_1(t) \pm x_2(t)], \frac{d}{dt}[y_1(t) \pm y_2(t)]\rangle$

(56)
(c) & (d) cont'd :

$$= \langle x_1'(t) \pm x_2'(t),\ y_1'(t) \pm y_2'(t) \rangle$$

$$= \langle x_1'(t), y_1'(t) \rangle \pm \langle x_2'(t), y_2'(t) \rangle$$

$$= \vec{r}_1{}'(t) \pm \vec{r}_2{}'(t)$$

as needed.

(e) Let $\vec{r}(t) = \langle x(t), y(t) \rangle$.

$$\frac{d}{dt}[f(t)\vec{r}(t)] = \frac{d}{dt}[\langle f(t)x(t), f(t)y(t) \rangle]$$

$$= \langle \frac{d}{dt}[f(t)x(t)], \frac{d}{dt}[f(t)y(t)] \rangle$$

$$= \langle f'(t)x(t) + f(t)x'(t),\ f'(t)y(t) + f(t)y'(t) \rangle$$

$$= \langle f(t)x'(t), f(t)y'(t) \rangle + \langle f'(t)x(t), f'(t)y(t) \rangle$$

$$= f(t)\langle x'(t), y'(t) \rangle + f'(t)\langle x(t), y(t) \rangle$$

$$= f(t)\vec{r}'(t) + f'(t)\vec{r}(t)$$

as needed.

(58) Prove Theorem 13.2.9 for $\mathbb{R}^2$.

Let $\vec{r}(t) = \langle x(t), y(t) \rangle$, $\vec{r}_1(t) = \langle x_1(t), y_1(t) \rangle$ & $\vec{r}_2(t) = \langle x_2(t), y_2(t) \rangle$.

(a) $$\int_a^b k\vec{r}(t)\,dt = \int_a^b k\langle x(t), y(t) \rangle\,dt$$

$$= \int_a^b \langle kx(t), ky(t) \rangle\,dt$$

(58) (a) contd:

$$= \left< \int_a^b kx(t)\,dt \,,\, \int_a^b ky(t)\,dt \right>$$

$$= \left< k\int_a^b x(t)\,dt,\ k\int_a^b y(t)\,dt \right>$$

$$= k \left< \int_a^b x(t)\,dt,\ \int_a^b y(t)\,dt \right>$$

$$= k \int_a^b \vec{r}(t)\,dt$$

as needed.

(b) & (c):

$$\int_a^b [\vec{r}_1(t) \pm \vec{r}_2(t)]\,dt = \int_a^b \left[\left< x_1(t), y_1(t) \right> \pm \left< x_2(t), y_2(t) \right> \right] dt$$

$$= \int_a^b \left< x_1(t) \pm x_2(t),\ y_1(t) \pm y_2(t) \right> dt$$

$$= \left< \int_a^b [x_1(t) \pm x_2(t)]\,dt,\ \int_a^b [y_1(t) \pm y_2(t)]\,dt \right>$$

$$= \left< \int_a^b x_1(t)\,dt \pm \int_a^b x_2(t)\,dt,\ \int_a^b y_1(t)\,dt \pm \int_a^b y_2(t)\,dt \right>$$

$$= \left< \int_a^b x_1(t)\,dt,\ \int_a^b y_1(t)\,dt \right> \pm \left< \int_a^b x_2(t)\,dt,\ \int_a^b y_2(t)\,dt \right>$$

$$= \int_a^b \vec{r}_1(t)\,dt \pm \int_a^b \vec{r}_2(t)\,dt$$

as needed.

Section 13.3

(32) $r = r(t)$, $\theta = \theta(t)$, $z = z(t)$

Know $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$

& $x = r(t)\cos(\theta(t))$; $y = r(t)\sin(\theta(t))$

So: $x' = r'\cos\theta - r(\sin\theta)\cdot\theta'$ (primes represent derivatives w/ respect to t)

$y' = r'\sin\theta + r(\cos\theta)\cdot\theta'$

$(x')^2 = (r')^2\cos^2\theta - 2rr'\cos\theta\sin\theta\,\theta' + r^2\sin^2\theta\,(\theta')^2$

$(y')^2 = (r')^2\sin^2\theta + 2rr'\sin\theta\cos\theta\,\theta' + r^2\cos^2\theta\,(\theta')^2$

$\Rightarrow (x')^2 + (y')^2 = (r')^2 + r^2(\theta')^2$

Thus

$$L = \int_a^b \sqrt{(r')^2 + r^2(\theta')^2 + (z')^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\cdot\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$ as needed.

(34) $\rho = \rho(t)$, $\theta = \theta(t)$, $\phi = \phi(t)$

We'll use the result from Exercise 32 & recall that $r = \rho(t)\sin(\phi(t))$ & $z = \rho(t)\cos(\phi(t))$, $\theta = \theta(t)$

So: $r' = \rho\cos\phi\cdot\phi' + \rho'\sin\phi$ (primes again represent derivatives w/respect to t)

$z' = -\rho\sin\phi\cdot\phi' + \rho'\cos\phi$

$(r')^2 = \rho^2\cos^2\phi\,(\phi')^2 + 2\rho\rho'\cos\phi\sin\phi\cdot\phi' + (\rho')^2\sin^2\phi$

$(z')^2 = \rho^2\sin^2\phi\,(\phi')^2 - 2\rho\rho'\cos\phi\sin\phi\cdot\phi' + (\rho')^2\cos^2\phi$

(34) $\Rightarrow (r')^2 + (z')^2 = \rho^2(\phi')^2 + (\rho')^2$

note: $r^2 = \rho^2 \sin^2\phi$

Thus $L = \int_a^b \sqrt{\rho^2(\phi')^2 + (\rho')^2 + \rho^2\sin^2\phi \cdot (\theta')^2}\, dt$

$$= \int_a^b \sqrt{\left(\frac{d\rho}{dt}\right)^2 + \rho^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + \rho^2\left(\frac{d\phi}{dt}\right)^2}\, dt$$

as needed.

(36) $\vec{r}(t) = t\vec{i} + t^2\vec{j} \quad t \in [-1, 1]$

(a)

$\vec{r}\,'(t) = \langle 1, 2t \rangle$: $\vec{r}\,'(t)$ is continuous & $\vec{r}\,'(t) \neq \vec{0}$ so $\vec{r}$ is <u>smooth</u>

Let $t = \tau^3$: $\vec{r}(\tau) = \langle \tau^3, \tau^6 \rangle$

$\vec{r}\,'(\tau) = \langle 3\tau^2, 6\tau^5 \rangle$ continuous

but $\vec{r}\,'(\tau) = \vec{0}$ at $\tau = 0$ $\Rightarrow$ $\vec{r}(\tau)$ <u>not</u> <u>smooth</u>

$\vec{r}(t)$: $\left.\begin{matrix} x = t \\ y = t^2 \end{matrix}\right\} y = x^2$ & $\vec{r}(\tau)$: $\left.\begin{matrix} x = \tau^3 \\ y = \tau^6 \end{matrix}\right\} y = x^2$

(same graph)

(b) The problem arises because $\frac{dt}{d\tau} = 3\tau^2 = 0$ when $t = 0$. By the chain rule:

$$\frac{d\vec{r}}{d\tau} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{d\tau}$$

If $t = \tau^3$: $\quad \frac{d\vec{r}}{d\tau} = \langle 1, 2t \rangle \cdot 3\tau^2$

& if $t = 0$, then $\tau = 0$, $\quad \left.\frac{d\vec{r}}{d\tau}\right|_{\tau=0} = \langle 1, 0 \rangle \cdot 0 = \vec{0}$ so that $\vec{r}$ is not smooth

In the first case: $\vec{r}(t) = \langle t, t^2 \rangle$

$\vec{r}\,'(t) = \langle 1, 2t \rangle$

& if $t = 0$, $\vec{r}\,'(0) = \langle 1, 0 \rangle$ so we do not have the same problem.