

Section 14.4

Proof:

⑥ For a linear function of 2 variables:

Let  $f(x,y) = ax + by + c$ . Then  $f_x = a$  &  $f_y = b$ .

$$\begin{aligned} \text{We have } \Delta f &= f(x+\Delta x, y+\Delta y) - f(x,y) \\ &= a(x+\Delta x) + b(y+\Delta y) + c - (ax + by + c) \\ &= a\Delta x + b\Delta y \end{aligned}$$

By Def 14.4.1:

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{a\Delta x + b\Delta y - a\Delta x - b\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} 0 = 0$$

$\therefore f(x,y) = ax + by + c$  is differentiable everywhere.

For a linear function of 3 variables:

Let  $f(x,y,z) = ax + by + cz + d$ . Then  $f_x = a$ ,  $f_y = b$  and  $f_z = c$ .

$$\begin{aligned} \text{We have } \Delta f &= f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z) \\ &= a(x+\Delta x) + b(y+\Delta y) + c(z+\Delta z) + d - (ax + by + cz + d) \\ &= a\Delta x + b\Delta y + c\Delta z \end{aligned}$$

By Def. 14.4.2:

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0 \\ \Delta z \rightarrow 0}} \frac{a\Delta x + b\Delta y + c\Delta z - a\Delta x - b\Delta y - c\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}$$

$$= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0 \\ \Delta z \rightarrow 0}} 0 = 0.$$

$\therefore f(x, y, z) = ax + by + cz + d$  is differentiable everywhere.

### Section 14.5

(74) Given:  $f, f_x, f_y$  continuous on a circular region containing  $A(x_0, y_0)$  &  $B(x_1, y_1)$ .

Show: there exists a point  $(x^*, y^*)$  on the line segment  $\overline{AB}$  s.t.  $f(x_1, y_1) - f(x_0, y_0)$

$$= f_x(x^*, y^*)(x_1 - x_0) + f_y(x^*, y^*)(y_1 - y_0)$$

Proof: Consider the line segment through  $A$  &  $B$ :

$$\vec{v} = \langle x_1 - x_0, y_1 - y_0 \rangle ; \text{ pt. } (x_0, y_0)$$

$$\Rightarrow \begin{cases} x = x_0 + (x_1 - x_0)t \\ y = y_0 + (y_1 - y_0)t \end{cases} \text{ for } t \in [0, 1]$$

Let  $F(t) = f(x(t), y(t))$ . Note  $F(0) = f(x_0, y_0) = f(A)$

&  $F(1) = f(x_1, y_1) = f(B)$ .

74 cont'd

Since  $F$  is continuous on  $[0, 1]$ , by MVT there exists  $t^* \in [0, 1]$  s.t.

$$F'(t^*) = \frac{F(1) - F(0)}{1 - 0} = f(x_1, y_1) - f(x_0, y_0)$$

Then for  $x^* = x(t^*)$ ,  $y^* = y(t^*)$  on  $\overline{AB}$   
We have:

$$f(x_1, y_1) - f(x_0, y_0) = F'(t^*)$$

$$= f_x \cdot x'(t) \Big|_{t^*} + f_y \cdot y'(t) \Big|_{t^*}$$

$$\begin{aligned} (x'(t) &= x_1 - x_0; \\ y'(t) &= y_1 - y_0) \end{aligned}$$

$$= f_x(x^*, y^*)(x_1 - x_0) + f_y(x^*, y^*)(y_1 - y_0)$$

as needed.

### Section 14.6

76 (for fn. of 2 variables)

Prove: If  $f, g$  are differentiable then:

$$(a) \quad \nabla(f+g) = \nabla f + \nabla g$$

$$\text{Proof: } \nabla(f+g) = \langle (f+g)_x, (f+g)_y \rangle$$

$$= \langle f_x + g_x, f_y + g_y \rangle$$

$$= \langle f_x, f_y \rangle + \langle g_x, g_y \rangle$$

$$= \nabla f + \nabla g$$

as needed.

$$(b) \nabla(cf) = c\nabla f$$

Proof: 
$$\begin{aligned} \nabla(cf) &= \langle (cf)_x, (cf)_y \rangle \\ &= \langle c \cdot f_x, c \cdot f_y \rangle \\ &= c \langle f_x, f_y \rangle \\ &= c \nabla f \quad \text{as needed.} \end{aligned}$$

$$(c) \nabla(fg) = f \nabla g + g \nabla f$$

Proof: 
$$\begin{aligned} \nabla(fg) &= \langle (fg)_x, (fg)_y \rangle \\ &= \langle f_x g + f g_x, f_y g + f g_y \rangle \\ &= \langle f g_x, f g_y \rangle + \langle f_x g, f_y g \rangle \\ &= f \langle g_x, g_y \rangle + g \langle f_x, f_y \rangle \\ &= f \nabla g + g \nabla f \quad \text{as needed.} \end{aligned}$$

$$(d) \nabla\left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2}$$

Proof: 
$$\begin{aligned} \nabla\left(\frac{f}{g}\right) &= \left\langle \left(\frac{f}{g}\right)_x, \left(\frac{f}{g}\right)_y \right\rangle \\ &= \left\langle \frac{g f_x - f g_x}{g^2}, \frac{g f_y - f g_y}{g^2} \right\rangle \\ &= \frac{1}{g^2} \left[ \langle g f_x, g f_y \rangle - \langle f g_x, f g_y \rangle \right] \\ &= \frac{1}{g^2} \left[ g \langle f_x, f_y \rangle - f \langle g_x, g_y \rangle \right] \\ &= \frac{g \nabla f - f \nabla g}{g^2} \quad \text{as needed.} \end{aligned}$$

76 cont'd

p.5

$$(e) \nabla f^n = n f^{n-1} \nabla f$$

Proof: 
$$\begin{aligned} \nabla f^n &= \langle (f^n)_x, (f^n)_y \rangle \\ &= \langle n f^{n-1} \cdot f_x, n f^{n-1} \cdot f_y \rangle \\ &= n f^{n-1} \langle f_x, f_y \rangle \\ &= n f^{n-1} \nabla f \end{aligned}$$
 as needed.

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Given:  $x = x(t)$ ,  $y = y(t)$  diff'ble at  $t$ ;  
 $z = f(x, y)$  diff'ble at  $(x(t), y(t))$

Show: 
$$\frac{dz}{dt} = \nabla z \cdot \vec{r}'(t) \quad \text{where} \quad \vec{r}(t) = \langle x(t), y(t) \rangle$$

Proof:

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \\ &= z_x \cdot x'(t) + z_y \cdot y'(t) \\ &= \langle z_x, z_y \rangle \cdot \langle x'(t), y'(t) \rangle \\ &= \nabla z \cdot \vec{r}'(t) \end{aligned}$$
 as needed.

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Prove: If  $f$  is diff'ble at  $(x, y)$  & if  
 $D_u f(x, y) = 0$  in 2 nonparallel directions,  
then  $D_v f(x, y) = 0$  in all directions.

Proof : (by contradiction)

Let  $\vec{u}_1 = \langle x_1, y_1 \rangle$  &  $\vec{u}_2 = \langle x_2, y_2 \rangle$  be non-parallel unit vectors for which  $D_{\vec{u}} f(x, y) = 0$ .

$$\text{Then } \nabla f \cdot \vec{u}_1 = f_x x_1 + f_y y_1 = 0$$

$$\& \nabla f \cdot \vec{u}_2 = f_x x_2 + f_y y_2 = 0.$$

Suppose  $\nabla f \neq \vec{0}$ . Then either  $f_x \neq 0$  or  $f_y \neq 0$ .

Assuming  $f_x \neq 0$ , we eliminate  $f_y$ :

$$-f_x x_1 y_2 - f_y y_1 y_2 = 0$$

$$f_x x_2 y_1 + f_y y_1 y_2 = 0$$

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$$f_x (x_2 y_1 - x_1 y_2) = 0$$

$$\Rightarrow x_2 y_1 - x_1 y_2 = 0$$

$$\Rightarrow x_2 y_1 = x_1 y_2$$

$$\Rightarrow \frac{x_2}{x_1} = \frac{y_2}{y_1}$$

$$\Rightarrow \vec{u}_1 \parallel \vec{u}_2 \quad \text{---} \times \text{---} \quad (\text{a contradiction})$$

Thus  $\nabla f = 0$  &  $D_{\vec{u}} f(x, y) = 0$  for all  $\vec{u}$ .