Exponential Growth and Decay

Exponential growth and decay refers to values or populations that expand or shrink by a fixed percentage rate, rather than at a fixed rate.

If you put \$100 in a bank account at 5% annual interest rate, you will at the end of the year have \$105. Leave the money in for another year and it will have increased by an additional 5%, to \$110.25.

Since the interest each year is calculated based on the amount of money in the account, the fixed interest rate yields a slightly greater amount each year.

At the end of any given year your total (***T***) can be expressed as:

$$T = P(1+R)^t \tag{1}$$

Where ***P*** is your principle or the original investment, ***R*** is your interest rate (expressed as a decimal, so 0.05 instead of 5%) and ***t*** is the total number of years (1, 2, 3, …).

Most banks do not calculate your interest annually however. If you calculate more often than once per year, the growth of your investment is greater; the growth of your total can be expressed as:

$$T = P(1+R/N)^{Nt} \tag{2}$$

Where ***N*** is the number of times per year that your interest is compounded.

N		***$P(1+R/N)^{Nt}$***	***= T***
1	(annual)	$T = 100(1+0.05)^t$	= \$105.00
2	(semi-annual)	$T = 100\ (1+0.05/2)^2$	= \$105.06
4	(quarterly)	$T = 100\ (1+0.05/4)^4$	= \$105.09
12	(monthly)	$T = 100\ (1+0.05/12)^{12}$	= \$105.12
365	(daily)	$T = 100\ (1+0.05/365)^{365}$	= \$105.13

As you can see, compounding interest more often leads to greater accumulation of money. Is there a limit to this? Let's experiment, using (for simplicity) ***R = 1*** and ***P = 1***:

N	***$P(1+R/N)^{Nt}$***	***= T***	
1	$(1+1/1)^1$	$= 2^1$	= 2
10	$(1+1/10)^{10}$	$= (1.1)^{10}$	= 2.59374246…
100	$(1+1/100)^{100}$	$= (1.01)^{100}$	= 2.70481383…
200	$(1+1/200)^{200}$	$= (1.005)^{100}$	= 2.711517123…
500	$(1+1/500)^{500}$	$= (1.002)^{100}$	= 2.715568521…
1,000	$(1+1/1000)^{1000}$	$= (1.001)^{1000}$	= 2.716923933…
10,000	$(1+1/10000)^{10000}$	$= (1.0001)^{10000}$	= 2.718145936…

Note that, as ***N*** becomes greater and greater, ***T*** continues to grow as well, but not nearly as much. The upper limit to the value of ***T*** is found when ***N*** becomes infinitely large; in this case it is a special number, known as ***e*** (named for the famous mathematician Euler):

e = 2.71828182846…

e, like ***π***, is a transcendental number, in that it is irrational (it cannot be expressed exactly as a fraction, or a ratio of two whole numbers), it cannot be calculated by a simple algebraic equation, and its decimal form consists of non-repeating numbers that go on forever.

so, for our exponential growth equation above we will define a number ***X = N/R***; by this definition ***R/N = 1/X*** *and* ***N = XR***. Plugging these values into the equation gives us:

$$T = P(1 + 1/X)^{XRt} \tag{2a}$$

We've already determined that as ***X*** becomes infinitely large, $(1+1/X)^X$ approaches ***e***, so we can redefine our equation thusly:

$$T = Pe^{Rt} \tag{3}$$

This equation allows us to calculate increasing interest on a bank account as well as an increasing population or even radioactive decay, since these all concern real-world values that compound continuously rather than at discrete intervals.

Suppose you have a population of bacteria that increase their population by ***R*** = 2% per minute (or 0.02/minute). There are ***P*** = 3000 of the bugs living in a Petri dish initially. How many will be there after ***t*** = 20 minutes?

$T = 3000e^{(0.02)\bullet(20)} = 4475$ bugs.

Because ***e*** is always used as an exponential base, it is used as the base of natural logarithms (abbreviated ***ln***), so that ***ln(5)*** is the number to which ***e*** must be raised to get 5:

$$e^{\ln(5)} = 5$$

$$e^{\ln(10)} = 10$$

So by the definition of exponents the following relationships are always true:

$$e^0 = 1 \qquad \ln(1) = 0$$

$$e^1 = e \qquad \ln(e) = 1$$

$$e^{-1} = 1/e \qquad \ln(1/e) = \ln(e^{-1}) = -1 \times \ln(e) = -1$$

$$e^{1/X} = \sqrt[X]{e} \qquad \ln(e^{1/X}) = (1/X) \times \ln(e) = 1/X$$

So how long before our original population of 3000 bugs doubles? In this case ***T = 2P***; if we plug this value into equation 3 we get:

$2P = Pe^{Rt_2}$; where t_2 is the time it takes the population to double.

Dividing both sides by ***P*** gives us:

$2 = e^{Rt_2}$; if we take the natural log of both sides we get:

$\ln(2) = \ln(e^{Rt_2}) = Rt_2 \times \ln(e) = Rt_2$; since ***ln(2) ~ 0.693*** we can generalize the solution:

$$t_2 = 0.693 / R \tag{4}$$

since, in this example, ***R = 0.02***, doubling time is 34.66 minutes. Since doubling time varies with the growth rate (***R***), this equation can be used to look at population dynamics.

Recall that the *percent growth rate* is the *decimal growth rate* multiplied by 100; if we use the *percent growth rate* in *equation 4* instead of the *decimal growth rate* we must also replace ***ln(2) ~ 0.693*** with***100*** *x* ***ln(2) = 69.3***. Rounding 69.3 to 70 we can now estimate the doubling time for a population with:

$$t_{doubling} = 70 / R \tag{4a}$$