Section 5.2 Volumes by Slicing: Disks and Washers

Pyramid:

We can see that the "slices" or "cross sections" are squares. The volume of each cross section is a rectangular solid where:

\[ V = \text{L} \cdot \text{W} \cdot \text{h} \]

We will use \( y = -x + 1 \) to model our pyramid

\[ V = 2x \cdot 2x \cdot \Delta y = 4x^2 \Delta y \]

Slice:

Volume: L \cdot W \cdot h
Since \( y = -x + 1 \)
\[ x = 1 - y \]

\[ V_{\text{slice}} = 4(1-y)^2 \Delta y \]

Now: Volume of pyramid

\[ V_{\text{pyramid}} = \int_0^1 4(1-y)^2 \, dy \]

Cone:

\[ V_{\text{Disk}} = \pi r^2 h \]

Since:

\[ y = -\frac{1}{2}x + 10 \]

\[ 5 \frac{1}{2} x = 10 - y \]

\[ x = \frac{2(10-y)}{5} \]

\[ x = \frac{20 - 2y}{5} \]

\[ V_{\text{Disk}} = \pi \left( \frac{20 - 2y}{5} \right)^2 \Delta y \]

\[ = \frac{\pi}{25} (20 - 2y)^2 \Delta y \]

Volume of Cone:

\[ V_{\text{cone}} = \frac{\pi}{25} \int_0^{10} (20 - 2y)^2 \, dy \]
Example 1: Derive the formula for the volume of a sphere.

\[ x^2 + y^2 = r^2 \]

\[ y^2 = r^2 - x^2 \]

\[ y = \sqrt{r^2 - x^2} \]

The radius of the disk is by the \( y \)-value.

\[ V_{\text{disk}} = \pi r^2 \cdot h \]

\[ V_{\text{disk}} = \pi y^2 \cdot \Delta x \]

\[ V_{\text{disk}} = \pi \left( \sqrt{r^2 - x^2} \right)^2 \Delta x \]

\[ = \pi \left( r^2 - x^2 \right) \Delta x \]

Since: \( y^2 = r^2 - x^2 \),
we can plug it in.

Volume of sphere:

\[ V_{\text{sphere}} = \int_{-r}^{r} \pi \left( r^2 - x^2 \right) dx \]

\[ = 2\pi \int_{0}^{r} \left( r^2 - x^2 \right) dx \]

\[ = 2\pi \left[ r^2x - \frac{x^3}{3} \right]_{0}^{r} \]

\[ = 2\pi \left[ \frac{2r^3}{3} - 0 \right] \]

\[ = \frac{4}{3}\pi r^3 \]
Example 2: Find the volume of the shape (Handout) having the circular base of radius 2, with cross-sections that are equilateral triangles.

\[ V_{\text{slice}} = \frac{1}{2} \cdot b \cdot h \cdot \Delta y \]

\[ = \frac{1}{2} \cdot (2y) \cdot \left( \frac{\sqrt{3}}{2} y \right) \Delta x \]

\[ = \frac{\sqrt{3}}{2} y^2 \Delta x \]

Since: \( x^2 + y^2 = 4 \)

\[ y^2 = 4 - x^2 \]

\[ \Rightarrow \frac{\sqrt{3}}{2} (4 - x^2) \Delta x \]

Volume:

\[ V = \int_{-2}^{2} \frac{\sqrt{3}}{2} (4 - x^2) \, dx \]
Example 3: Find the volume of a shape whose base is a semi-circle of radius 2 whose cross-sections are isosceles triangles where the base is equal to the height.

\[ V_{\text{slice}} = \frac{1}{2} \cdot (y)(y) \Delta x \]

\[ = \frac{1}{2} \cdot y^2 \Delta x \]

\[ V_{\text{slice}} = \frac{1}{2} (4-x^2) \Delta x \]

\[ V = \int_{-2}^{2} \frac{1}{2} (4-x^2) \, dx \]

\[ = 2 \left( \frac{1}{2} \right) \int_{0}^{2} (4-x^2) \, dx \]

\[ \Rightarrow V = \int_{0}^{2} (4-x^2) \, dx \]
Example 4: Find the volume of a shape whose base is a circle of radius 2, where cross-sections are squares.

Since: \( y = \sqrt{4-x^2} \)

\[ V_{\text{slice}} = 4 \left( \sqrt{4-x^2} \right)^2 \Delta x \]
\[ = 4 (4-x^2) \Delta x \]

\[ V = \int_{-2}^{2} (4-x^2) \Delta x \]

\[ V = 2 \cdot 4 \int_{0}^{2} (4-x^2) \Delta x \]

\[ = 8 \int_{0}^{2} (4-x^2) \, dx \]
\[ = 8 \left[ 4x - \frac{x^3}{3} \right]_0^2 \]
\[ = 8 \left[ (8 - \frac{8}{3}) - 0 \right] \]
\[ = 8 \left[ \frac{24}{3} - \frac{8}{3} \right] \]
\[ = 8 \left( \frac{16}{3} \right) \]
\[ = \frac{128}{3} \]
Example 5: Set up the integral that represents the volume of the shaped form by rotating the bounded area about the indicated axis.

a) \( y = x^2 \), \( x = 0 \), \( x = 4 \) : \( x \)-axis

\[
V_{\text{Disk}} = \pi r^2 \Delta x
\]
\[
= \pi y^2 \Delta x
\]
\[
= \pi [x^2]^2 \Delta x
\]

\[
V = \pi \int_{0}^{4} x^4 \, dx
\]

b) \( y = x^2 \), \( x = 0 \), \( x = 4 \) : \( y \)-axis

\[
y = x^2 \rightarrow x = \sqrt{y}
\]

\[
R = 4
\]
\[
\Gamma = x = \sqrt{y}
\]

\[
V_{\text{Washer}} = \pi \left[ R^2 - r^2 \right] \Delta y
\]
\[
= \pi \left[ y^2 - (\sqrt{y})^2 \right] \Delta y
\]

\[
V = \pi \int_{0}^{16} \left[ 16 - y \right] \, dy
\]
c) $y = x^2$, $x = 0$, $x = 4$: $y$-axis

$$V_{\text{shell}} = 2\pi rh \cdot \Delta x$$

$h = y = x^2$

$r = x$

$$V_{\text{shell}} = 2\pi x \cdot x^2 \cdot \Delta x$$

$$V = 2\pi \int_0^4 x^3 \, dx$$

d) $y = x^2$, $y = x$: $x$-axis

$$V_{\text{washer}} = \pi [R^2 - r^2] \Delta x$$

$R = y = x$

$r = y = x^2$

$$x = \pi [(x)^2 - (x^2)^2] \Delta x$$

$$V = \pi \int_0^4 [x^2 - x^4] \, dx$$
\( f(x) = -\frac{1}{4} (x - 4)^2 + 18 \)

\( g(x) = 2 \)

\( x = 1 \quad \text{X-axis} \)

\( x = 8 \)

\[ R = y = -\frac{1}{4} (x - 4)^2 + 18 \]

\[ r = y = 2 \]

\[ V_{\text{washer}} = \pi \int [R^2 - r^2] \, \Delta x \]

\[ = \pi \int \left[ \left( -\frac{1}{4} (x - 4)^2 + 18 \right)^2 - (2)^2 \right] \, \Delta x \]

\[ V = \pi \int_{1}^{8} \left\{ \left[ -\frac{1}{4} (x - 4)^2 + 18 \right]^2 - 4 \right\} \, dx \]
(f) \( X = y^2 + 1 \), \( X = y + 2 \), Rotate about \( x = 5 \)

Point of Intersection:
\[ y^2 = y + 2 \]
\[ y^2 - y - 2 = 0 \]
\[ (y - 2)(y + 1) = 0 \]
\[ y = 2 \quad y = -1 \]
\[ x = 4 \quad x = 1 \]

\[ R = 5 - x_1 \quad \text{Different } x \text{ values!} \]
\[ r = 5 - x_2 \]

\[ R = 5 - y^2 \]
\[ r = 5 - (y + 2) \]

\[ V_{\text{washer}} = \pi \int [R^2 - r^2] \, dy \]
\[ = \pi \int [(5 - y^2)^2 - (5 - (y + 2))^2] \, dy \]

\[ V = \pi \int_{-1}^{2} [(5 - y^2) - (3 - y)^2] \, dy \]