Derivatives of Trig Functions
Selected Problems

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Derivatives of Trig Functions: Selected Problems

1. Find $f'(x)$:

   (a) $f(x) = 4 \cos x + 2 \sin x$

   $f'(x) = 4 \frac{d}{dx}(\cos x) + 2 \frac{d}{dx}(\sin x)$
   $= -4 \sin x + 2 \cos x$

   (b) $f(x) = 2 \sin^2 x = 2 (\sin x)^2 = 2 \sin x \sin x$

   $f'(x) = 2 \frac{d}{dx}(\sin x \sin x)$
   $= 2(\sin x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(\sin x))$
   $= 2(\sin x \cos x + \sin x \cos x)$
   $= 2(2 \sin x \cos x)$
   $= 4 \sin x \cos x$

   (c) $f(x) = \sec x - \sqrt{2} \tan x$

   $f'(x) = \frac{d}{dx}(\sec x) - \sqrt{2} \frac{d}{dx}(\tan x)$
   $= \sec x \tan x - \sqrt{2} \sec^2 x$
(d) \[ f(x) = \frac{5 - \cos x}{5 + \sin x} \]

\[ f'(x) = \frac{(5 + \sin x) \frac{d}{dx}(5 - \cos x) - (5 - \cos x) \frac{d}{dx}(5 + \sin x)}{(5 + \sin x)^2} \]

\[ = \frac{(5 + \sin x)(\sin x) - (5 - \cos x)(\cos x)}{(5 + \sin x)^2} \]

\[ = \frac{5\sin x + \sin^2 x - 5\cos x + \cos^2 x}{(5 + \sin x)^2} \]

\[ = \frac{(\sin^2 x + \cos^2 x) + 5(\sin x - \cos x)}{(5 + \sin x)^2} \]

\[ = \frac{1 + 5(\sin x - \cos x)}{(5 + \sin x)^2} \]

(e) \[ f(x) = \sec x \tan x \]

\[ f'(x) = \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \]

\[ = \sec x(\sec^2 x) + \tan x(\sec x \tan x) \]

\[ = \sec^2 x + \sec x \tan^2 x \]
2. Find $\frac{d^2y}{dx^2}$:

(a) $y = x \cos x$

\[
\frac{dy}{dx} = \cos x \frac{d}{dx}(x) + x \frac{d}{dx}(\cos x)
\]
\[
= \cos x - x \sin x
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dx}(\cos x) - \frac{d}{dx}(x \sin x)
\]
\[
= -\sin x - \left(\sin x \frac{d}{dx}(x) + x \frac{d}{dx}(\sin x)\right)
\]
\[
= -\sin x - (\sin x + x \cos x)
\]
\[
= -\sin x - \sin x - x \cos x
\]
\[
= \boxed{-2 \sin x - x \cos x}
\]

(b) $y = \csc x$

\[
\frac{dy}{dx} = \frac{d}{dx}(\csc x)
\]
\[
= -\csc x \cot x
\]

\[
\frac{d^2y}{dx^2} = -\frac{d}{dx}(\csc x \cot x)
\]
\[
= -(\cot x \frac{d}{dx}(\csc x) + \csc x \frac{d}{dx}(\csc x))
\]
\[
= -(\cot x (-\csc x \cot x) + \csc x (-\csc^2 x))
\]
\[
= \boxed{\cot^2 x \csc x + \csc^3 x}
\]
3. Find the equation of the line tangent to the graph of \( \sin x \) at \( x = \pi/4 \):

Let \( f(x) = \sin x \), then \( f\left(\frac{\pi}{4}\right) = \sin \left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \).

Now \( f'(x) = \cos x \), so \( f'\left(\frac{\pi}{4}\right) = \cos \left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \).

We now have a point, \( \left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \), and a slope, \( \frac{1}{\sqrt{2}} \). So plugging this into the point-slope formula we obtain our line:

\[
y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left( x - \frac{\pi}{4} \right)
\]

\[
y - \frac{1}{\sqrt{2}} = \frac{x}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}
\]

\[
y = \frac{x}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}
\]

4. Show that \( y = \cos x \) and \( y = \sin x \) are solutions to the equation \( y'' + y = 0 \).

\[
y = \cos x
\]

\[
y' = -\sin x
\]

\[
y'' = -\cos x
\]

\[
y'' + y = -\cos x + \cos x = 0.
\]

\[
y = \sin x
\]

\[
y' = \cos x
\]

\[
y'' = -\sin x
\]

\[
y'' + y = -\sin x + \sin x = 0.
\]
5. A 10-ft ladder leans against a wall at an angle $\theta$. The top of the ladder is $x$ feet above the ground. If the bottom of the ladder is pushed towards the wall, find the rate at which $x$ changes with respect to $\theta$ when $\theta = 60^\circ$. Express the answer in units of feet/degree.

$$
\sin \theta = \frac{x}{10}
$$

$$
x = 10 \sin \theta
$$

$$
\frac{dx}{d\theta} = 10 \frac{d}{d\theta} (\sin \theta)
$$

$$
= 10 \cos \theta
$$

$$
\left. \frac{dx}{d\theta} \right|_{\theta=60^\circ}=10 \cos \left( \frac{\pi}{3} \right) = 10 \left( \frac{1}{2} \right) = 5
$$

$$
= \frac{5 \text{ ft}}{\text{rad}} \left( \frac{\pi \text{ rad}}{180^\circ} \right)
$$

$$
= \frac{\pi \text{ ft}}{36^\circ} \approx 0.087 \text{ ft/deg}
$$
6. An airplane is flying on a horizontal path at a height of 3800 ft. At what rate is the distance $s$ between the airplane and the point $P$ changing with respect to $\theta$ when $\theta = 30^\circ$? Express the answer in units of feet/degree.

\[
\sin \theta = \frac{3800}{s} \\
s = \frac{3800}{\sin \theta} = 3800 \csc \theta
\]

\[
\frac{ds}{d\theta} = 3800 \frac{d}{d\theta} (\csc \theta) \\
= -3800 \csc \theta \cot \theta
\]

\[
\left. \frac{ds}{d\theta} \right|_{\theta=30^\circ=\frac{\pi}{6}} = -3800 \csc \left( \frac{\pi}{6} \right) \cot \left( \frac{\pi}{6} \right) \\
= -3800(2)(\sqrt{3}) = -7600\sqrt{3} \\
= -7600\sqrt{3} \frac{\text{rad}}{\text{ft}} \left( \frac{\pi \text{ rad}}{180^\circ} \right) \\
= -\frac{380\sqrt{3}\pi}{9} \frac{\text{ft}}{\text{deg}} \approx -230 \frac{\text{ft}}{\text{deg}}
\]
7. Determine where $f$ is differentiable.

(a) $f(x) = \sin x$

$\sin x$ is differentiable for all real values of $x$ because it is continuous for all real values of $x$.

(b) $f(x) = \cot x$

$\cot x = \frac{\cos x}{\sin x}$ is not continuous whenever $\sin x = 0$, or $x = 0, \pi, 2\pi, 3\pi, \ldots$

So $f(x)$ is differentiable for all $x \neq n\pi$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

(c) $f(x) = \frac{1}{1+\cos x}$

This is not continuous whenever

$1 + \cos x = 0$

$\cos x = -1$

$x = \pi, 3\pi, 5\pi, \ldots$

So $f(x)$ is differentiable for all $x \neq (2n + 1)\pi$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$