Section 2.9 Local Linear Approximations

Differentials

Visual Representation:

Δy is referred to as the "propagated Error".

Note that the tangent line passes through \((x_0, f(x_0))\)

\[
\therefore m = \tan f'(x_0)
\]

Now we find the equation of the tangent line using

\[
y - y_1 = m(x - x_1)
\]

\[
y - f(x_0) = f'(x_0)(x - x_0)
\]

\[
y = f'(x_0)(x - x_0) + f(x_0)
\]

This equation of the tangent line of \(f(x)\) at \(x_0\) is called "The Local Linear Approximation" of \(f(x)\) in the vicinity of \(x_0\).

We therefore can say:

\[
f(x) \approx f'(x_0)(x - x_0) + f(x_0)
\]

This is a \(y\)-value on \(f(x)\)

This is a \(y\)-value on the tangent line of \(f(x)\) at \(x_0\)
Example 1: Find the Local Linear Approximation
for \( f(x) = \sqrt{x} \) at \( x_0 = 9 \).

Then approximate \( \sqrt{9.1} \) using the
Local Linear Approximation

\[
f(x) = \sqrt{x} \quad \Rightarrow \quad f'(x) = \frac{1}{2\sqrt{x}}
\]

\[
f'(9) = \frac{1}{6}
\]

The Local Linear Approximation:

\[
y = f'(9)(x-9) + f(9)
\]

\[
y = \frac{1}{6}(x-9) + 3
\]

\[
y = \frac{1}{6}x - \frac{3}{2} + \frac{6}{2}
\]

\[
* \quad y = \frac{1}{6}x + \frac{3}{2}
\]

We can now say that \( f(x) = \sqrt{x} \) is approximately

\[
\frac{1}{6}x + \frac{3}{2}
\]

\[
\Rightarrow \quad f(x) \approx \frac{1}{6}x + \frac{3}{2} \quad \text{in the vicinity of \( x = 9 \)}
\]

Note: \( f(9.1) \approx \frac{1}{6} \left( \frac{91}{10} \right) + \frac{3}{2} \)

\[
= \frac{1}{6} \left( \frac{91}{10} \right) + \frac{3}{2}
\]

\[
= \frac{91}{60} + \frac{90}{2}
\]
\( V = \pi r^2 h \)
\( V = 15\pi r^2 \)
\( \frac{d}{dr} [V] = \frac{d}{dr} [15\pi r^2] \)
\( \frac{dV}{dr} = 30\pi r \)
\( dV = 30\pi r \cdot dr \)
\( dV = 30\pi \left(\frac{5}{2}\right) \cdot \left(\frac{1}{10}\right) \)
\( = 30\pi \cdot \left(\frac{1}{4}\right) \)
\( = \frac{30\pi}{4} \Rightarrow \frac{15\pi}{2} \text{ cm}^3 \left(7.5\pi \text{ cm}^3\right) \)

This is a "local linear approximation."

Exact value of the volume of insolation:
\( V_{\text{with}} = 15\pi \left(2.6\right)^2 = 318.5674 \text{ cm}^3 \)

\( V_{\text{without}} = 15\pi \left(2.5\right)^2 = 294.5243 \text{ cm}^3 \)

Insolation volume: \( V_{\text{with}} - V_{\text{without}} = 24.0331 \text{ cm}^3 \) (exact)