Concave up

→ The slopes of the tangent lines are increasing. Therefore $f'(x)$ is increasing.
$f''(x) > 0$

Concave down

→ The slopes of the tangent lines are decreasing. Therefore $f'(x)$ is decreasing.
$f''(x) < 0$

Section 3.2: Relative Extrema + Graphing Polynomials

Definition:

$\begin{align*}
\text{Relative Max at } x_0^+ & : f(x_0) > f(x) \\
& \quad f'(x_0) = 0 \\
& \quad f''(x_0) < 0
\end{align*}$

$\begin{align*}
\text{Relative Min at } x_0^- & : f(x_0) < f(x) \\
& \quad f'(x_0) = 0 \\
& \quad f''(x_0) > 0
\end{align*}$

Relative Maximums and Relative Minimums are referred to as "Relative Extremum"
Theorem 3.2.2

Suppose that \( f(x) \) is defined on an open interval containing \( x_0 \). If \( f \) has a relative extremum at \( x_0 \), then \( x = x_0 \) is called a "critical point." This means \( f'(x_0) = 0 \) or \( f'(x_0) = \text{undefined} \).

\[ \begin{align*}
  &f'(x_0) = 0 \\
  &f'(x_0) = \text{undefined}
\end{align*} \]

Note: If \( x_0 \) is a critical point, and \( f''(x_0) = 0 \), we classify \( x_0 \) as a "stationary point."

Theorem 3.2.3 (First Derivative Test)

If \( f'(x) \) changes from positive to negative this implies a Relative Maximum.

If \( f'(x) \) changes from negative to positive this implies a Relative Minimum.

Note: \( f(x) \) is continuous and \( x_0 \) is a critical point.

If \( f'(x) \) does not change signs then there is no Relative Extremum.
Theorem 3.2.4 (The Second Derivative Test)

Suppose that \( f(x) \) is twice differentiable at \( x_0 \).
- If \( f'(x_0) = 0 \) and \( f''(x_0) > 0 \), then there is a Relative Minimum at \( x_0 \).
- If \( f'(x_0) = 0 \) and \( f''(x_0) < 0 \), then there is a Relative Maximum at \( x_0 \).
- If \( f'(x_0) = 0 \) and \( f''(x_0) = 0 \), then: Inconclusive

Definition 3.2.5

→ The Geometric Interpretation of Multiplicity

Suppose that \( p(x) \) is a polynomial function with a root of multiplicity \( "m" \) at \( x = r \).

\[ p(x) = (x - r)^m \]

a) If \( "m" \) is even, then the graph of \( f(x) \) is tangent to the \( x \)-axis at \( x = r \).

b) If \( "m" \) is odd and \( > 1 \), then the graph is tangent to the \( x \)-axis, crosses the \( x \)-axis, and \( x_0 \) is an inflection point.

c) If \( "m" = 1 \), then the graph is not tangent to the \( x \)-axis, crosses the \( x \)-axis, and may or may not be an inflection point.
Example 1:

Inflection point

Increasing

This has to be positive
#9 Find the critical points and identify which critical points are stationary points.

\[ f(x) = \frac{x + 1}{x^2 + 3} \]

\[ f'(x) = \frac{(1)(x^2 + 3) - (x + 1)(2x)}{(x^2 + 3)^2} \]

\[ = \frac{x^2 + 3 - 2x^2 - 2}{(x^2 + 3)^2} \]

\[ = \frac{-x^2 - 2x + 3}{(x^2 + 3)^2} \]

\[ = -\frac{(x^2 + 2x - 3)}{(x^2 + 3)^2} \]

\[ = -\frac{(x + 3)(x - 1)}{(x^2 + 3)^2} \]

Setting \( f'(x) = 0 \)

\[ (x + 3) = 0 \]
\[ x = -3 \]

\[ (x - 1) = 0 \]
\[ x = 1 \]

Critical points: \( x = -3 \), \( x = 1 \)

Stationary points: \( x = -3 \), \( x = 1 \)

#29 Find any relative extremum using \( f'(x) + f''(x) \)

\[ f(x) = 1 + 8x - 3x^2 \]

\[ = -3x^2 + 8x + 1 \]

\[ f'(x) = -6x + 8 \]

\[ f''(x) = -6 \]

Set \( f'(x) = 0 \)

\[ -6x + 8 = 0 \]
\[ x = \frac{4}{3} \]

Evaluating \( f''(x) \) when \( x = \frac{4}{3} \):

\[ f''\left(\frac{4}{3}\right) = -6 \]

Relative Maximum at \( x = \frac{4}{3} \).