SECTION: THE DEFINITE INTEGRAL:

RIEMAN SUMS:

Previously we approximate area with the equal widths.

In this case we do not use equal widths.

\[ \sum_{k=1}^{n} f(x_k^*) \Delta x \] is replaced by regular partition.

\[ \sum_{k=1}^{n} f(x_k^*) \Delta x_k \]

NOTE: If the partitions have equal widths, the partition is said to be regular.

- Allowing unequal widths replaces \( \sum_{k=1}^{n} f(x_k^*) \Delta x \) with \( \sum_{k=1}^{n} f(x_k^*) \Delta x_k \).
- Using \( \Delta x_k \), we let \( \max \Delta x_k \to 0 \) which has the same effect as letting \( n \to \infty \).
- Therefore we have

\[ A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k \]

DEFINITION: 4.5.1:

If the limit exists, we say that \( f \) is integrable on \([a, b]\):

\[ \int_{a}^{b} f(x) \, dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k \]

DEFINITE INTEGRAL RIEMANN SUM.
THEOREM

\[ A = \int_{a}^{b} f(x) \, dx \] represents the net signed area between \( f(x) \) and \( y = 0 \).

\[ \int_{0}^{2\pi} \sin x \, dx = 0 \]

EXAMPLE 1: Evaluate the integral by graphing \( f \) and using an approximate geometric formula.

(a) \[ \int_{-1}^{1} \sqrt{1 - x^2} \, dx \]

Notice: \( y = \sqrt{1 - x^2} \)

\[ y^2 = 1 - x^2 \]

\[ y^2 + x^2 = 1 \]

\[ \int_{-1}^{1} \sqrt{1 - x^2} \, dx = \frac{\pi}{2} \]

(b) \[ \int_{0}^{2} (x - 1) \, dx = 0 \]

DEFINITION:

\[ \int_{a}^{a} f(x) \, dx = 0 \]

\[ \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \]

THEOREM

\[ \int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx \]

\[ \int_{a}^{b} [f(x) + g(x)] \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \]

THEOREM

\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \]
**Example 2:**

\[ \int_{1}^{2} |2x - 3| \, dx \]

\[ |2x - 3| = \begin{cases} 
2x - 3 & x \geq \frac{3}{2} \\
-x + 3 & x \leq \frac{3}{2} 
\end{cases} \]

\[ A_1 = \frac{1}{2} \left( \frac{5}{2} \right) (5) \]

\[ A_1 = \frac{25}{4} \]

\[ A_2 = \frac{1}{2} \left( \frac{1}{2} \right) 1 \]

\[ A_2 = \frac{1}{4} \]

\[ \int_{1}^{2} |2x - 3| \, dx = \frac{25}{4} + \frac{1}{4} = \frac{26}{4} = \frac{13}{2} \]

**Example 3:**

\[ \int_{0}^{1} (x + 2 \sqrt{1-x^2}) \, dx \]

\[ \int_{0}^{1} x \, dx + 2 \int_{0}^{1} \sqrt{1-x^2} \, dx \]

\[ A_1 = \frac{1}{2} (1)(1) \]

\[ A_1 = \frac{\pi}{4} \]

\[ = \frac{1}{2} \]

\[ A_T = \frac{1}{2} + 2 \left( \frac{\pi}{2} \right) = \frac{1 + \pi}{2} \]