**CHAPTER 2: DERIVATIVES**

**Derivative** - the slope of the tangent line and their rates of change.

**SECTION 2.1 TANGENT LINES & RATES OF CHANGE**

Finding Equations of Tangent Lines

Equation of a line:

\[ y - y_1 = m(x - x_1) \]

\[ y - f(x_0) = m(x - x_0) \]

Recall:

\[ m_{\text{sec}} = \lim_{x_1 \to x_0} \left[ \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] \]

\[ m_{\text{tan}} = \lim_{x_1 \to x_0} \left[ \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] \]

\[ m_{\text{sec}} = \frac{f(x_0 + h) - f(x_0)}{h} \]

\[ m_{\text{tan}} = \lim_{h \to 0} \left[ \frac{f(x_0 + h) - f(x_0)}{h} \right] \]
Find the equation of the tangent line on \( y = \frac{2}{x} \), at \((2,1)\).

\[
\left[ x, f(x) \right] = (2, 1)
\]

\[
m_{\text{tan}} = \lim_{h \to 0} \left[ \frac{f(x+h) - f(x)}{h} \right]
\]

\[
\lim_{h \to 0} \left[ \frac{2}{2h} - 1 \right]
\]

\[
\lim_{h \to 0} \left[ \frac{2}{2h} - 1 \right] \cdot \frac{2h}{2h}
\]

\[
\lim_{h \to 0} \left[ \frac{2(2h) \left( \frac{2h}{1} \right) - 1 \left( \frac{2h}{1} \right)}{h(2h)} \right]
\]

\[
\lim_{h \to 0} \left[ \frac{2-(2+h)}{h(2h)} \right]
\]

\[
\lim_{h \to 0} \left[ \frac{2-2-h}{h(2h)} \right]
\]

\[
\lim_{h \to 0} \left[ \frac{-h}{h(2h)} \right]
\]

\[
\lim_{h \to 0} \left[ \frac{-1}{2h} \right]
\]

\[
-\frac{1}{2} = \frac{1}{2}
\]

This value represents the derivative of \( f(x) \) at \( x = 2 \).

\[ m_{\text{tan}} = -\frac{1}{2} \,, \quad (2,1) \]

\[ y - y_1 = m (x - x_1) \]

\[ y - 1 = -\frac{1}{2} (x - 2) \]

\[ y - 1 = -\frac{1}{2} x + 1 \]

\[ y = -\frac{1}{2} x + 2 \quad \text{Eqln of Tan Line} \]
SECTION 2.1 cont...

E1 cont. Suppose we want a formula that would give us the slope of the tangent line for any value of x.

\[ f(x) = \frac{2}{x} \]

\[ \lim_{{h \to 0}} \left[ \frac{f(x+h) - f(x)}{h} \right] \] "The Limit Definition of the Derivative"

\[ \lim_{{h \to 0}} \left[ \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \right] = \frac{x(x+h)}{x(x+h)} \]

\[ \lim_{{h \to 0}} \left[ \frac{2x - 2(x+h)}{hx(x+h)} \right] \]

\[ \lim_{{h \to 0}} \left[ \frac{2x - 2x - 2h}{hx(x+h)} \right] \]

\[ \lim_{{h \to 0}} \left[ \frac{-2h}{hx(x+h)} \right] \]

\[ \lim_{{h \to 0}} \left[ \frac{-2}{x(x+h)} \right] \]

\[ \frac{-2}{x(x+h)} = \frac{2}{x(x)} = \frac{-2}{x^2} \]

Note: derivative notation

\[ f'(x) = -\frac{2}{x^2} \]

"f" prime of x = first derivative

\[ f''(x) = \text{second derivative} \]

E2. The position function.

What is the average speed over the 4 seconds?

\[ m = \frac{20 \text{ ft}}{4 \text{ sec}} \]

\[ m = 5 \text{ ft/sec} \]
[E3] Use the graph to answer the following questions.

- maximum velocity in the positive direction.
- maximum velocity in the negative direction.

\[ M_{\text{sec}} = \frac{2}{3} \text{ cm/sec}. \]

(Seconds)

a) When is the instantaneous velocity equal to \( \varnothing \)?

0, 2, 4, 8

b) What is \( V_{\text{avg}} \) on \([0, 8]\)?

0

c) When is \( V_{\text{inst.}} \) at maximum?

1, 2 sec. positive
3.1 sec negative

NOTE

\[ M_{\text{tan}} = \text{instantaneous} \]
\[ M_{\text{sec}} = \text{average} \]
SECTION: TANGENT LINES & RATES OF CHANGE

\[ y - y_1 = m(x - x_1) \]

- \( y - y_1 = m(x - x_1) \)
  - tangent line: \( m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \)
  - tangent line: \( m_{\text{tan}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \)

- EQUATION OF TANGENT LINE
  \[ y - f(x_0) = m_{\text{tan}} (x - x_0) \]

\[ m_{\text{sec}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \]

**NOTE:**

\[ m_{\text{tan}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \]

\[ m_{\text{sec}} = \frac{f(x_0 + h) - f(x_0)}{h} \]

\[ m_{\text{tan}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \]

- **EXAMPLE 1:** Find the equation of the tangent line on \( y = \frac{2}{x} \)

  at \( (2, 1) \)

  \[ m_{\text{tan}} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} \]

  \[ = \lim_{h \to 0} \frac{\frac{2}{2+h} - 1}{-h} \frac{(2+h) - 1}{h} \frac{2-h}{h(2+h)} \]
Example 1 continues:

\[ m_{\text{tan}} = \lim_{h \to 0} \frac{-1}{2 + h} \]

\[ m_{\text{tan}} = -\frac{1}{2} \]

\[ -\frac{1}{2} \text{ is } m_{\text{tan}} \text{ at } x = 2 \text{ on } y = \frac{2}{x} \]

\[ m_{\text{tan}} = -\frac{1}{2} \quad (2, 1) \]

\[ y - f(x_0) = m_{\text{tan}}(x - x_0) \]

\[ y - 1 = -\frac{1}{2}(x - 2) \]

\[ y = -\frac{x}{2} + 2 \]

Suppose we wanted a formula that would give us the slope of the tangent for any value of \( x \)

\[ m_{\text{tan}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \]

Limit definition of the derivative of \( f(x) \) \( [f'(x) \text{ "f prime of } x\text{"}] \)

Where \( f'(x) \) gives you the slope of the tangent line on \( f(x) \) at any given value of \( x \)!

Example 2: Find \( f'(x) \) for \( f(x) = \frac{2}{x} \)

\[ f'(x) = \lim_{h \to 0} \left[ \frac{\frac{2}{x + h} - \frac{2}{x}}{h} \right] x(x + h) \]

\[ = \lim_{h \to 0} \frac{2x - 2(x + h)}{h(x + h)} \]

\[ = \lim_{h \to 0} \frac{-2h}{h(x + h)} \]

\[ = \frac{-2}{x(x + h)} \]

\[ f'(x) = -\frac{2}{x^2} \]

Note: \( f'(2) = -\frac{1}{2} \quad i 

f'(3) = -\frac{2}{9} \quad i 

f'(4) = -\frac{1}{8} \]
**Example 3:**

a) average velocity over the interval $0 \leq t \leq 3$

b) values of $t$ at which the instantaneous velocity is zero

$$t = 0, 2, 4, 6$$

c) values of $t$ at which the instantaneous velocity is either max or min

$$\text{max: } t = 1$$
$$\text{min: } t = 3$$

d) instantaneous velocity when $t = 3$.

$$v(t) = \frac{15 \text{ cm}}{(3.5 - 2.3) \text{ sec}} \approx -10 \text{ cm/sec}$$

**Example 4** $y = 2x^2$, $x_0 = 0$, $x_1 = 1$

a) $V_{avg} = \frac{2}{1} = m_{sec}$

b) $V_{inst} = \frac{0}{0} = m_{tan}$

c) $m_{tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$

$$= \lim_{h \to 0} \frac{2(x_0 + h)^2 - 2x_0}{h}$$
**Example 4 continues:**

c) \[ \lim_\limits_{h \to 0} \frac{2x^2 + 4x\cdot h + 2h^2 - 2x^2}{h} \]

\[ = \lim_\limits_{h \to 0} \frac{h(4x_0 + 2h)}{h} \]

\[ = 4x_0 + 2(0) \]

\[ m_{\tan} = 4x_0 \]

**Example 5:** \( f(x) = x^2 - 1 \quad x_0 = 1 \)

we want the equation of the tangent line on \( f(x) = x^2 - 1 \quad x_0 = -1 \)

\[ m_{\tan} = \lim_\limits_{h \to 0} \frac{(x_0 + h)^2 - 1 - (x_0^2 - 1)}{h} \]

\[ = \lim_\limits_{h \to 0} \frac{x_0^2 + 2x_0h + h^2 - x_0^2 + 1}{h} \]

\[ = \lim_\limits_{h \to 0} \frac{h(2x_0 + h)}{h} \]

\[ = 2x_0 \]

\[ \therefore \text{the } m_{\tan} @ x_0 = -1 \text{ is } -2 \]

since \( m_{\tan} = -2 \) the line passes through \((-1,0)\) we get \( y - y_1 = m(x - x_1) \)

\[ y - 0 = 2(x - (-1)) \Rightarrow y = -2(x + 1) \]