Calculus I
Sample Exam #03

1. Integrate the following. Rationalize the denominator if necessary and write your solution as a single fraction.

   a) \( \int_{\pi/6}^{\pi/3} \sin x \cos x \, dx \)

   b) \( \int_{\pi/6}^{2\pi/3} \frac{\cos x}{\sin^2 x} \, dx \)
2. Integrate the following. Rationalize the denominator if necessary and write your solution as a single fraction.

a) \[ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \tan x \sec^2 x \, dx \]

b) \[ \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos^2 x} \, dx \]
3. Evaluate the integral by making the appropriate substitutions. Write your solution as a single fraction.

\[ a) \int_{\frac{4\pi^2}{9}}^{\frac{16\pi^2}{9}} \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx \]

\[ b) \int_0^3 \frac{x^2}{\sqrt{x+1}} \, dx \]
4. Given, \( \int_{-2}^{3} f(x) \, dx = -2 \), \( \int_{-2}^{3} g(x) \, dx = -4 \), \( \frac{2}{3} \int_{-2}^{3} h(x) \, dx = 1 \), find the following:

a) \( 2 \int_{-2}^{3} [g(x) - 2x] \, dx \) 

b) \( \int_{-2}^{3} \left[ 2f(x) - \frac{3}{2} h(x) \right] \, dx \)

5. A particle moves along the s-axis. Use the given information to find the position function of the particle.

\[
a(t) = 1 + \cos 2t ; \quad v(\pi) = 0 ; \quad s \left( \frac{\pi}{4} \right) = \frac{\pi^2}{32}
\]
6. A particle moves with acceleration \( a(t) \, m/s^2 \) along an s-axis and has velocity \( v_0 \, m/s \) at time \( t = 0 \). Find the displacement and the distance traveled by the particle during the stated time interval.

\[ a(t) = 6t - 4; \quad v(0) = 1; \quad 0 \leq t \leq 1 \]
7. A solid has a circular base with equation $x^2 + y^2 = 4$. Parallel cross-sections perpendicular to the y-axis are equilateral triangles. Find the volume.
8. A solid has a circular base with equation \( x^2 + y^2 = 4 \). Parallel cross-sections perpendicular to the x-axis are rectangles whose height is twice the length of its width. Find the volume.
9. The tank below is full of heating oil with density 100 lb/ft$^3$. Find the work required to pump the oil out the outlet. See figure.
10. The region shown in the figure to the right is bounded by the graphs:

\[ y = e^{-x}, \ y = \sqrt{x+1}, \ x = 0 \text{ and } x = 1 \]

Write the integral (DO NOT INTEGRATE!) for the volume \( V \) of the solid obtained by revolving the region about the stated line. Use the indicated method.

a) \( y = -1 \); Washer Method

b) \( x = -1 \); Shell Method

c) \( y = 3 \); Washer Method

d) \( x = 2 \); Shell Method
11. The flat surface shown is submerged in a fluid. Find the fluid force against the surface. The fluid density is $10 \text{ lbs/ft}^3$. 

\[ \text{Force} = \rho \cdot g \cdot A \cdot h \] 

where $\rho$ is the fluid density, $g$ is the acceleration due to gravity, $A$ is the area of the surface, and $h$ is the depth of the fluid.
Calculus I
Sample Exam #03

1. Integrate the following. Rationalize the denominator if necessary and write your solution as a single fraction.

a) \( \int_{\pi/6}^{\pi/3} \sin x \cos x \, dx \)

\[ u = \sin x \]
\[ du = \cos x \, dx \]
\[ x = \frac{\pi}{6} \rightarrow u = \sin \frac{\pi}{6} = \frac{1}{2} \]
\[ x = \frac{\pi}{3} \rightarrow u = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \]

\[ \int_{1/2}^{\sqrt{3}/2} u \, du \]

\[ = \frac{u^2}{2} \bigg|_{1/2}^{\sqrt{3}/2} \]

\[ = \frac{1}{2} \left[ \left( \frac{\sqrt{3}}{2} \right)^2 - \left( \frac{1}{2} \right)^2 \right] \]

\[ = \frac{1}{2} \left[ \frac{3}{4} - \frac{1}{4} \right] \]

\[ = \frac{1}{4} \left[ \frac{2}{4} \right] \]

\[ = \frac{1}{4} \]

b) \( \int_{\pi/6}^{2\pi/3} \frac{\cos x}{\sin^2 x} \, dx \)

\[ u = \sin x \]
\[ du = \cos x \, dx \]
\[ x = \frac{2\pi}{3} \rightarrow u = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \]
\[ x = \frac{\pi}{6} \rightarrow u = \sin \frac{\pi}{6} = \frac{1}{2} \]

\[ \int_{1/2}^{\sqrt{3}/2} \frac{1}{u^2} \, du \]

\[ = \int_{1/2}^{\sqrt{3}/2} u^{-2} \, du \]

\[ = - \frac{1}{u} \bigg|_{1/2}^{\sqrt{3}/2} \]

\[ = - \frac{1}{\frac{\sqrt{3}}{2}} - \left( - \frac{1}{\frac{1}{2}} \right) \]

\[ = -2 + 2 \]

\[ = \frac{-2\sqrt{3}}{3} + \frac{6}{3} \]

\[ = \frac{0 - 2\sqrt{3}}{3} \]
2. Integrate the following. Rationalize the denominator if necessary and write your solution as a single fraction.

a) \[ \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x \sec^2 x \, dx \]

\[ u = \tan x \]
\[ du = \sec^2 x \, dx \]
\[ x = \frac{\pi}{3} \rightarrow u = \tan \frac{\pi}{3} = \sqrt{3} \]
\[ x = \frac{\pi}{4} \rightarrow u = \tan \frac{\pi}{4} = 1 \]
\[ \int_{1}^{\sqrt{3}} u \, du \]
\[ = \frac{u^2}{2} \bigg|_{1}^{\sqrt{3}} \]
\[ = \frac{(\sqrt{3})^2}{2} - \frac{1^2}{2} \]
\[ = \frac{3}{2} - \frac{1}{2} \]
\[ = \frac{2}{2} \]
\[ = 1 \]

b) \[ \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \frac{\sin x}{\cos^2 x} \, dx \]

\[ u = \cos x \]
\[ du = -\sin x \, dx \]
\[ -du = \sin x \, dx \]
\[ x = \frac{2\pi}{3} \rightarrow u = \cos \frac{2\pi}{3} = -\frac{1}{2} \]
\[ x = \frac{\pi}{6} \rightarrow u = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \]
\[ -\int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} u^{-2} \, du \]
\[ = \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} u^{-2} \, du \]
\[ = -\frac{1}{u} \bigg|_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \]
\[ = \left( -\frac{2}{\sqrt{3}} \right) - \left( \frac{2}{\sqrt{3}} \right) \]
\[ = -2\frac{\sqrt{3}}{3} - \frac{6}{3} \]
\[ = \frac{-2\sqrt{3} - 6}{3} \]
3. Evaluate the integral by making the appropriate substitutions. Write your solution as a single fraction.

a) \[ \int_{4\pi^2}^{16\pi^2} \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx \]

\[ U = \sqrt{x} \frac{4}{1} \]
\[ du = \frac{1}{2} \sqrt{x} \, dx \]
\[ 2 \, du = \frac{1}{\sqrt{x}} \, dx \]
\[ x = \frac{4\pi^2}{9} \rightarrow u = \frac{2\pi}{3} \]
\[ x = \frac{16\pi^2}{9} \rightarrow u = \frac{4\pi}{3} \]

\[ 2 \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \sec^2 u \, du \]

\[ = 2 \tan u \bigg|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \]
\[ = 2 \left[ \tan \frac{4\pi}{3} - \tan \frac{2\pi}{3} \right] \]
\[ = 2 \left[ \sqrt{3} - (-\sqrt{3}) \right] \]
\[ = 2 \left[ 2\sqrt{3} \right] \]
\[ = 4\sqrt{3} \]

b) \[ \int_{0}^{3} \frac{x^2}{\sqrt{x+1}} \, dx \]

\[ U = x + 1 \]
\[ x = u - 1 \]
\[ x^2 = u^2 - 2u + 1 \]
\[ du = dx \]
\[ x = 0 \rightarrow u = 1 \]
\[ x = 3 \rightarrow u = 4 \]

\[ \int_{1}^{4} \frac{u^2 - 2u + 1}{\sqrt{u}} \, du \]

\[ = \int_{1}^{4} (u^{3/2} - 2u^{1/2} + u^{-1/2}) \, du \]
\[ = \frac{2}{5} u^{5/2} - \frac{4}{3} u^{3/2} + 2u^{1/2} \bigg|_{1}^{4} \]
\[ = \left( \frac{2}{5} \cdot 32 - \frac{4}{3} \cdot 8 + 2 \cdot 2 \right) - \left( \frac{2}{5} - \frac{3}{3} + 2 \right) \]
\[ = \frac{64}{5} - \frac{32}{3} + 4 - \frac{2}{5} + \frac{4}{3} - 2 \]
\[ = \frac{62}{5} - \frac{28}{3} + 2 \]
\[ = \frac{186}{15} - \frac{140}{15} + \frac{30}{15} \]
\[ = \frac{46}{15} + \frac{30}{15} \]
\[ = \frac{76}{15} \]
4. Given, \( \int_{-2}^{3} f(x) \, dx = -2 \), \( \int_{-2}^{3} g(x) \, dx = -4 \), \( \frac{2}{3} \int_{-2}^{3} h(x) \, dx = 1 \), find the following:

   a) \( 2 \int_{-2}^{3} [g(x) - 2x] \, dx \)

\[
2 \int_{-2}^{3} g(x) \, dx - 2 \int_{-2}^{3} 2x \, dx
= 2(-4) - 2 \left[ x^2 \right]_{-2}^{3}
= -8 - 2 \left[ 9 - 4 \right]
= -8 - 2(5)
= -8 - 10
= -10
\]

   b) \( \frac{3}{2} \int_{-2}^{3} h(x) \, dx = \frac{3}{2} \int_{-2}^{3} f(x) \, dx - \frac{3}{2} \int_{-2}^{3} h(x) \, dx \)

\[
2 \int_{-2}^{3} f(x) \, dx - \frac{3}{2} \int_{-2}^{3} h(x) \, dx
= 2(-2) - \frac{3}{2} \left( \frac{3}{2} \right)
= -4 - \frac{9}{4}
= -\frac{16}{4} - \frac{9}{4}
= -\frac{25}{4}
\]

5. A particle moves along the \( s \)-axis. Use the given information to find the position function of the particle.

\[
v(t) = \int a(t) \, dt
\]

\[
a(t) = 1 + \cos 2t ; \quad v(\pi) = 0 ; \quad s(\pi/4) = \frac{\pi^2}{32}
\]

\[
v(t) = \int \left( t + \frac{1}{2} \sin 2t \right) \, dt
\]

\[
0 = \pi + \frac{1}{2} \sin 2\pi + C
\]

\[
-\pi = C
\]

\[
v(t) = t + \frac{1}{2} \sin 2t - \pi
\]

\[
a(t) = \frac{\pi^2}{2} - \frac{\pi}{4} \cos \frac{\pi t}{2} - \pi \left( \frac{\pi}{4} \right) + C
\]

\[
\frac{\pi^2}{32} = \frac{\left( \frac{\pi}{4} \right)^2}{2} - \frac{1}{4} \cos \frac{\pi}{2} - \pi \left( \frac{\pi}{4} \right) + C
\]

\[
\frac{\pi^2}{32} = \frac{\pi^2}{32} - 0 - \frac{\pi^2}{4} + C
\]

\[
\frac{\pi^2}{4} = C
\]

\[
s(t) = \frac{t^2}{2} - \frac{\pi}{4} \cos 2t - \pi t + \frac{\pi^2}{4}
\]
6. A particle moves with acceleration \( a(t) \text{ m/s}^2 \) along an \( s \)-axis and has velocity \( v_0 \text{ m/s} \) at time \( t = 0 \). Find the displacement and the distance traveled by the particle during the stated time interval.

\[ a(t) = 6t - 4; \quad v(0) = 1; \quad 0 \leq t \leq 1 \]

\[
V(t) = \int a(t) \, dt \\
V(t) = \int (6t - 4) \, dt \\
V(\phi) = 3t^2 - 4t + C \\
1 = 3(\phi)^2 - 4(\phi) + C \\
1 = C \\
V(t) = 3t^2 - 4t + 1 \\
V(t) = (3t - 1)(t - 1)
\]

\[
S(t) = \int_{0}^{1} (3t^2 - 4t + 1) \, dt \\
S(t) = \left[ t^3 - 2t^2 + t \right]_{0}^{1/3} \\
= (1^3 - 2(1/3)^2 + 1/3) - (0^3) \\
= \frac{1}{27} - \frac{2}{9} + \frac{1}{3} - 0 \\
= \frac{4}{27}
\]

\[
S(t) = \int_{\phi}^{1/3} (3t^2 - 4t + 1) \, dt \\
S(t) = \left[ t^3 - 2t^2 + t \right]_{\phi}^{1/3} \\
= (1^3 - 2(1/3)^2 + 1/3) - (\phi^3) \\
= (1 - 2 + 1) - (\phi^3) \\
DISTANCE = -\frac{4}{27}
\]

\[
\int_{0}^{1} |V(t)| \, dt = \left| \frac{4}{27} \right| + \left| -\frac{4}{27} \right| = \frac{8}{27} \text{ m}
\]

\[
\int_{\phi}^{1/3} v(t) \, dt = \frac{4}{27} + (-\frac{4}{27}) = 0 \text{ m}
\]

\[
\text{DISTANCE: } \frac{8}{27} \text{ m} \\
\text{DISPLACEMENT: } 0 \text{ m}
\]
7. A solid has a circular base with equation $x^2 + y^2 = 4$. Parallel cross-sections perpendicular to the y-axis are equilateral triangles. Find the volume.

$V = 2 \int_0^2 \left( \sqrt{3} y - \sqrt{3} y^2 \right) dy$

$V = 2 \left[ 4\sqrt{3} y - \frac{\sqrt{3}}{3} y^3 \right]_0^2$

$V = 2 \left[ 4\sqrt{3}(2) - \frac{\sqrt{3}}{3}(8) - 0 \right]$

$V = 16\sqrt{3} - \frac{16\sqrt{3}}{3}$

$V = \frac{32\sqrt{3}}{3}$
8. A solid has a circular base with equation $x^2 + y^2 = 4$. Parallel cross-sections perpendicular to the x-axis are rectangles whose height is twice the length of its width. Find the volume.

\[ V = l \cdot w \cdot h \]
\[ V = 2y \cdot \Delta x \cdot 4y \]
\[ V = 8y^2 \Delta x \]
\[ V = 8(4 - x^2) \Delta x \]
\[ V = 8 \cdot 2 \int_0^2 (4 - x^2) \, dx \]
\[ V = 16 \left[ 4x - \frac{x^3}{3} \right]_0^2 \]
\[ V = 16 \left[ 8 - \frac{8}{3} - 0 \right] \]
\[ V = 16 \left[ \frac{24}{3} - \frac{8}{3} \right] \]
\[ V = 16 \left( \frac{16}{3} \right) \]
\[ V = \frac{256}{3} \]
9. The tank below is full of heating oil with density 100 lb/ft³. Find the work required to pump the oil out the outlet. See figure.

\[ V = lwh \]
\[ V = 5 \cdot 2x \cdot \Delta y \]
\[ V = 10x \Delta y \]
\[ V = - \frac{10}{3} y \Delta y \text{ ft}^3 \]

**WEIGHT = \rho \cdot Volume**

\[ = 1000 \left( -\frac{10}{3} y \right) \Delta y \]

\[ = - \frac{10000}{3} y \Delta y \text{ lbs} \]

**WORK = \int \rho \cdot volume \cdot distance**

\[ = - \frac{10000}{3} \int_{3}^{0} y(2-y) \, dy \]

\[ = - \frac{10000}{3} \int_{3}^{0} 2y - y^2 \, dy \]

\[ = - \frac{10000}{3} \left[ y^2 - \frac{y^3}{3} \right]_{0}^{3} \]

\[ = - \frac{10000}{3} \left[ 0 - \left( 9 + \frac{27}{3} \right) \right] \]

\[ = - \frac{10000}{3} (-18) \]

\[ = 60000 \text{ ft} \cdot \text{lbs} \]
10. The region shown in the figure to the right is bounded by the graphs:
\( y = e^{-x}, \ y = \sqrt{x+1}, \ x = 0 \) and \( x = 1 \)
Write the integral (DO NOT INTEGRATE!) for the volume \( V \) of the solid obtained by revolving the region about the stated line. Use the indicated method.

a) \( y = -1; \) Washer Method
\[
\int_{0}^{1} \pi \left[ (1 + \sqrt{x+1})^2 - (1 + e^{-x})^2 \right] dx
\]

b) \( x = -1; \) Shell Method
\[
\int_{0}^{1} 2\pi (1 + x)(\sqrt{x+1} - e^{-x}) dx
\]

c) \( y = 3; \) Washer Method
\[
\int_{0}^{1} \pi \left[ (3 - e^{-x})^2 - (3 - \sqrt{x+1})^2 \right] dx
\]

d) \( x = 2; \) Shell Method
\[
\int_{0}^{1} 2\pi (2 - x)(\sqrt{x+1} - e^{-x}) dx
\]
11. The flat surface shown is submerged in a fluid. Find the fluid force against the surface. The fluid density is \(10 \text{ lbs/ft}^3\). \textbf{Note:} \(F = \int_a^b \rho h(x) w(x) \, dx\)

\[
X^2 + Y^2 = 16
\]

\[
X = \sqrt{16 - Y^2}
\]

\[
F = \rho h(Y) w(Y) \Delta Y
\]

\[
= 10(Y - 4)(2 \Delta Y)
\]

\[
F = 10(Y - 4)(2 \sqrt{16 - Y^2}) \Delta Y
\]

\[
F = 20 \int_{-4}^{4} (Y - 4)(\sqrt{16 - Y^2}) \, dy
\]

\[
F = 20 \int_{-4}^{4} 6\sqrt{16 - Y^2} \, dy - 20 \int_{-4}^{4} Y\sqrt{16 - Y^2} \, dy
\]

\[
\frac{1}{2} du = \gamma \, dy
\]

\[
\gamma = 4 \rightarrow u = 0
\]

\[
\gamma = -4 \rightarrow u = 0
\]

\[
F = 120 \int_{-4}^{4} \sqrt{16 - Y^2} \, dy
\]

\[
= 120 (\theta \pi)
\]

\[
F = 960 \pi \text{ lbs}
\]
1. Integrate the following. Rationalize the denominator if necessary and write your solution as a single fraction.

   a) $\int_1^2 \frac{6x - 3x^2}{\sqrt{3x^2 - x^3}} \, dx$

   b) $\int_6^9 \frac{x - 3}{\sqrt{x - 5}} \, dx$
2. Evaluate the integral by making the appropriate substitutions. Write your solution as a single fraction.

a) \[ \int_{\frac{3}{\pi}}^{\frac{18}{\pi}} \frac{\sec^2\left(\frac{3}{x}\right)}{x^2} \, dx \]

b) \[ \int_{\frac{9}{\pi}}^{\frac{\pi}{12}} \sin^2 3x \cos 3x \, dx \]
3. Divide the specified interval into \( n = 4 \) subintervals of equal length and then compute
\[
\sum_{k=1}^{4} f \left( x^*_k \right) \Delta x
\]
with \( x^*_k \) as the **midpoint** of the subinterval. **Write your solution as a single fraction.**

\[
f(x) = \cos x \quad \left[ 0, \frac{4\pi}{3} \right]
\]

4. Find the exact arc length of the curve \( y = 3x^{3/2} - 1 \) from \( x = 0 \) to \( x = 1 \). Write your solutions as a single fraction.

\[
L = \int_{a}^{b} \sqrt{1 + \left[ f'(x) \right]^2} \, dx
\]
5. A particle moves with acceleration $a(t) \text{ m/s}^2$ along an $s$-axis and has velocity $v_0 \text{ m/s}$ at time $t = 0$. Find the displacement and the distance traveled by the particle during the given time interval.

$a(t) = -8t + 12; \quad v(0) = -8; \quad -2 \leq t \leq 2$
6. A solid has an elliptical base with equation \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \). Parallel cross-sections perpendicular to the x-axis are squares. Find the volume.
7. The tank below is filled to a depth of 6 feet with heating oil of density 10 lb/ft³. Find the work required to pump the oil out the outlet. See figure.
8. The region shown in the figure to the right is bounded by the graphs:
\[ y = \sqrt{x} + 3, \ y = 3, \ x = 1 \text{ and } x = 4 \]
Write the integral for the volume \( V \) of the solid obtained by revolving the region about the stated axis (DO NOT INTEGRATE!).

a) \( x \)-axis; Washer Method

b) \( y \)-axis; Washer Method

c) \( y = -1; \) Shell Method

d) \( x = 5; \) Shell Method
9. The flat surface shown is submerged vertically in a fluid. It is a rectangle with two semicircular ends.

Find the fluid force against the surface. The fluid density is \(10 \frac{\text{lbs}}{\text{ft}^3}\).

**Note:** \(F = \int_a^b \rho h(y) w(y) \, dy\)
Math3A
Exam #03 Solutions
Fall 2016

1. Integrate the following. Rationalize the denominator if necessary and write your solution as a single fraction.

a) \[ \int_{2}^{3} \frac{6x^2 - x^3}{\sqrt{3x^2 - x^3}} \, dx \]

\[ u = 3x^2 - x^3 \]
\[ du = (6x - 3x^2) \, dx \]
\[ \begin{align*}
  x = 2 & \implies u = 3(2)^2 - 2^3 \\
  u = 12 - 8 &= 4 \\
  x = 1 & \implies u = 3(1)^2 - 1^3 \\
  u = 3 - 1 &= 2
\end{align*} \]

\[ \int_{4}^{2} \frac{1}{\sqrt{u}} \, du \]
\[ \int_{2}^{4} u^{-\frac{1}{2}} \, du \]
\[ 2u^{\frac{1}{2}} \bigg| _2^4 \\
 2 \left[ 4^{\frac{1}{2}} - 2^{\frac{1}{2}} \right] \\
 2 \left[ 2 - \sqrt{2} \right] \\
\]

b) \[ \int_{6}^{9} \frac{x - 3}{\sqrt{x - 5}} \, dx \]

\[ u = x - 5 \implies x = u + 5 \]
\[ du = dx \]
\[ x = 9 \implies u = 9 - 5 = 4 \]
\[ x = 6 \implies u = 6 - 5 = 1 \]

\[ \int_{4}^{1} \frac{u + 5 - 3}{\sqrt{u}} \, du \]
\[ \int_{1}^{4} \frac{u + 2}{\sqrt{u}} \, du \]
\[ \int_{1}^{4} \left[ u^{\frac{1}{2}} + 2u^{-\frac{1}{2}} \right] \, du \]
\[ \frac{2}{3}u^{\frac{3}{2}} + 4u^{\frac{1}{2}} \bigg| _1^4 \\
\]
\[ \left[ \frac{2}{3}(8) + 4(2) \right] - \left[ \frac{2}{3} + 4 \right] \\
\frac{16}{3} + \frac{24}{3} - \frac{2}{3} - \frac{12}{3} \]
\[ \frac{26}{3} \]
2. Evaluate the integral by making the appropriate substitutions. Write your solution as a single fraction.

\[ \int_{\frac{9}{12}}^{\frac{\pi}{12}} \sin^2 3x \cos 3x \, dx \]

\[ u = \sin 3x \]
\[ du = 3 \cos 3x \, dx \]
\[ \frac{1}{3} du = \cos 3x \, dx \]
\[ x = \frac{4}{3} \Rightarrow u = \sin \left[ 3 \left( \frac{4}{3} \right) \right] = \sin \frac{4}{3} = \frac{\sqrt{2}}{2} \]
\[ x = \frac{2}{3} \Rightarrow u = \sin \left[ 3 \left( \frac{2}{3} \right) \right] = \sin \frac{2}{3} = \frac{\sqrt{3}}{3} \]

\[ \frac{1}{3} \int_{\frac{\pi}{3}}^{1} u^2 \, du \]
\[ \frac{1}{3} \cdot \frac{u^3}{3} \bigg|_{\frac{\sqrt{2}}{2}}^{1} \bigg|_{\frac{\sqrt{3}}{3}} \]
\[ \frac{1}{9} \left[ 1^3 - (\frac{\sqrt{2}}{2})^3 \right] \]
\[ \frac{1}{9} \left[ 1 - \frac{2\sqrt{2}}{8} \right] \]
\[ \frac{1}{9} \left[ 4 - \frac{\sqrt{12}}{4} \right] \]
\[ \frac{1}{9} \left[ 4 - \frac{\sqrt{12}}{4} \right] \]
\[ \frac{4 - \sqrt{12}}{36} \]

\[ \frac{3}{x^2} \]
\[ \int_{\frac{9}{12}}^{\frac{\pi}{12}} \sec^2 \left( \frac{3}{x} \right) \, dx \]

\[ u = \frac{3}{x} \]
\[ du = -\frac{3}{x^2} \, dx \]
\[ -\frac{1}{3} du = \frac{1}{x^2} \, dx \]
\[ x = \frac{18}{\pi} \Rightarrow u = \frac{3}{\frac{18}{\pi}} = \frac{\pi}{6} \]
\[ x = \frac{9}{\pi} \Rightarrow u = \frac{3}{\frac{9}{\pi}} = \frac{\pi}{3} \]

\[ -\frac{1}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 u \, du \]
\[ \frac{1}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 u \, du \]
\[ \frac{1}{3} \tan u \bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \]
\[ \frac{1}{3} \left[ \tan \left( \frac{\pi}{3} \right) - \tan \left( \frac{\pi}{6} \right) \right] \]
\[ \frac{1}{3} \left[ \sqrt{3} - \frac{1}{\sqrt{3}} \right] \]
\[ \frac{1}{3} \left[ \frac{3\sqrt{3}}{3} - \frac{\sqrt{3}}{3} \right] \]
\[ \frac{1}{3} \left[ 2\sqrt{3} \right] \]
\[ \frac{2\sqrt{3}}{9} \]
3. Divide the specified interval into \( n = 4 \) subintervals of equal length and then compute \( \sum_{i=1}^{4} f(x_i^*) \Delta x \) with \( x_i^* \) as the midpoint of the subinterval. Write your solution as a single fraction.

\[
\Delta x = \frac{b-a}{n} = \frac{\frac{4\pi}{3} - 0}{4} = \frac{\pi}{3} \\
\sum_{i=1}^{4} f(x_i^*) \Delta x = \frac{\pi}{3} \left[ f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right) + f\left(\frac{5\pi}{6}\right) + f\left(\frac{7\pi}{6}\right) \right] \\
= \frac{\pi}{3} \left[ \cos \frac{\pi}{6} + \cos \frac{\pi}{3} + \cos \frac{5\pi}{6} + \cos \frac{7\pi}{6} \right] \\
= \frac{\pi}{3} \left[ \frac{\sqrt{3}}{2} + 0 + \left( -\frac{\sqrt{3}}{2} \right) + \left( -\frac{\sqrt{3}}{2} \right) \right] \\
= \frac{\pi}{3} \left[ -\frac{\sqrt{3}}{2} \right] \\
= -\frac{\sqrt{3} \pi}{6}
\]

4. Find the exact arc length of the curve \( y = 3x^{3/2} - 1 \) from \( x = 0 \) to \( x = 1 \). Write your solutions as a single fraction. \( L = \int_{0}^{1} \sqrt{1 + [f'(x)]^2} \, dx \)

\[
f(x) = 3x^{3/2} - 1 \\
f'(x) = \frac{9}{2} x^{1/2} \\
L = \int_{0}^{1} \sqrt{1 + \left( \frac{9}{2} x^{1/2} \right)^2} \, dx \\
= \int_{0}^{1} \sqrt{1 + \frac{81}{4} x} \, dx \\
= \int_{0}^{1} \sqrt{\frac{4 + 81x}{4}} \, dx \\
= \frac{1}{81} \int_{0}^{85} u^{1/2} \, du \\
= \frac{1}{162} \cdot \frac{2}{3} \left[ u^{3/2} \right]_{4}^{85} \\
= \frac{1}{243} \left[ 85^{3/2} - 4^{3/2} \right] \\
= \frac{85\sqrt{85} - 8}{243}
\]
5. A particle moves with acceleration \( a(t) \text{ m/s}^2 \) along an \( s \)-axis and has velocity \( v_0 \text{ m/s} \) at time \( t = 0 \). Find the displacement and the distance traveled by the particle during the given time interval.

\[ a(t) = -8t + 12; \quad v(0) = -8; \quad -2 \leq t \leq 2 \]

\[
\begin{align*}
V(t) &= \int a(t) \, dt \\
&= \int (-8t + 12) \, dt \\
&= -4t^2 + 12t + C
\end{align*}
\]

Applying the initial condition \( v(0) = -8 \):

\[
-8 = -4(0)^2 + 12(0) + C
\]

\[
-8 = C
\]

\[
V(t) = -4t^2 + 12t - 8
\]

\[
V(t) = -4(t^2 - 3t + 2)
\]

\[
V(t) = -4(t - 2)(t - 1)
\]

**Displacement:**

\[
\begin{align*}
\text{Disp.} &= \int_{-2}^{2} V(t) \, dt \\
&= \int_{-2}^{2} (-4t^2 + 12t - 8) \, dt \\
&= -\frac{4}{3}t^3 + 6t^2 - 8t \bigg|_{-2}^{2} \\
&= -\frac{4}{3}(8) + 6(4) - 8(2) - \left(-\frac{4}{3}(-8) + 6(4) - 8(-2)\right) \\
&= \frac{-32}{3} + 24 - 16 - \frac{32}{3} - 24 + 16 \\
&= -\frac{64}{3} - 32 \\
&= -\frac{64}{3} - \frac{96}{3} \\
&= \boxed{-\frac{160}{3} \text{ m}}
\end{align*}
\]

**Distance:**

\[
\begin{align*}
\text{Distance} &= \int_{-2}^{2} |V(t)| \, dt \\
&= \int_{-2}^{1} (4t^2 - 12t + 8) \, dt + \int_{1}^{2} (-4t^2 + 12t - 8) \, dt \\
&= \frac{4}{3}t^3 - 6t^2 + 8t \bigg|_{-2}^{1} + \left(-\frac{4}{3}t^3 + 6t^2 - 8t\right) \bigg|_{1}^{2} \\
&= \frac{4}{3} - 6 + 8 - \left(-\frac{32}{3} - 24 - 16\right) - \frac{32}{3} + 24 - 16 - \left(-\frac{4}{3} + 6 - 8\right) \\
&= \frac{4}{3} + 2 + \left(-\frac{32}{3} + 24 + 16 + \frac{32}{3} + 8 + \frac{4}{3} - 6 + 8\right) \\
&= \frac{2}{3} + 58 - 6 \\
&= \frac{8}{3} + 52 \\
&= \frac{8}{3} + \frac{156}{3} \\
&= \boxed{\frac{164}{3} \text{ m}}
\end{align*}
\]
6. A solid has an elliptical base with equation \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \). Parallel cross-sections perpendicular to the x-axis are squares. Find the volume.

\[
\begin{align*}
\sqrt{y} &= \int_{-2}^{2} 9(4-x^2) \, dx \\
&= 18 \int_{0}^{2} (4-x^2) \, dx \\
&= 18 \left[ 4x - \frac{x^3}{3} \right]_{0}^{2} \\
&= 18 \left[ 8 - \frac{8}{3} - 0 \right] \\
&= 18 \left[ \frac{16}{3} \right] \\
&= 6 \cdot 16 \\
&= \boxed{96}
\end{align*}
\]
7. The tank below is filled to a depth of 6 feet with heating oil of density 10 lb/ft³. Find the work required to pump the oil out the outlet. See figure.

\[ d = 2 - y \]

\[ y = -\frac{9}{2} x \]

\[-\frac{9}{2} y = x \]

\[ W = \int_{a}^{b} F(y) \, dy \]

\[ = \int_{-9}^{-3} 10 \left( \frac{800}{9} \right) (2-y) \, dy \]

\[ = \frac{800}{9} \int_{-9}^{-3} y (y-2) \, dy \]

\[ = \frac{800}{9} \left[ \frac{y^3}{3} - y^2 \right]_{-9}^{-3} \]

\[ = \frac{800}{9} \left[ \left( \frac{-3}{3} \right) - \left( \frac{-9}{3} \right) \right] \]

\[ = \frac{800}{9} \left[ \frac{-18 + 243 + 81}{3} \right] \]

\[ = \frac{800}{9} \left[ 81 \right] \]

\[ = 800(34) \]

\[ = 27,200 \text{ ft} \cdot \text{lbs} \]
8. The region shown in the figure to the right is bounded by the graphs;
\[ y = \sqrt{x} + 3, \ y = 3, \ x = 1 \text{ and } x = 4 \]
Write the integral for the volume \( V \) of the solid obtained by revolving the region about the stated axis (DO NOT INTEGRATE!).

\[ V = \pi \int_1^4 \left[ (\sqrt{x}+3)^2 - (3)^2 \right] \, dx \]

b) \( y = 1 \); Washer Method
\[ V = \pi \int_3^4 \left[ (\sqrt{x}-1)^2 - 1^2 \right] \, dy + \pi \int_4^5 \left[ 4^2 - (y-3)^2 \right] \, dy \]

c) \( y = -1 \); Shell Method
\[ V = 2\pi \int_3^4 (y+4)(4-x) \, dy + 2\pi \int_4^5 (y+1)(4-(y-3)^2) \, dy \]

d) \( x = 5 \); Shell Method
\[ V = 2\pi \int_1^4 (5-x)(\sqrt{x}+3-3) \, dx \]
\[ V = 2\pi \int_1^4 (5-x)\sqrt{x} \, dx \]
9. The flat surface shown is submerged vertically in a fluid. It is a rectangle with two semicircular ends.

Find the fluid force against the surface. The fluid density is \(10 \frac{\text{lbs}}{\text{ft}^3}\).

Note: \(F = \int_0^b \rho h(y) w(y) \, dy\)

\[ F = \int_{-1}^{1} \left[ 10 \cdot (2-y) \left( 2\sqrt{1-y^2} + 6 \right) \right] \, dy \]

\[ F = 20 \int_{-1}^{1} \left[ (2-y) \left( \sqrt{1-y^2} + 3 \right) \right] \, dy \]

\[ F = 20 \int_{-1}^{1} \left[ \frac{2(1-y^2)}{2} + 6 - y \sqrt{1-y^2} - 3y \right] \, dy \]

\[ A = \frac{1}{2} \pi (4)^2 \quad \text{Odd Function} \]

\[ A = \frac{\pi}{2} \]

\[ F = 20 \left[ 2\left(\frac{\pi}{2}\right) + 6 \right] \left|_{-1}^{1} \right. - 0 - \frac{3y^2}{2} \left|_{-1}^{1} \right. \]

\[ F = 20 \left[ \pi + (6-(-6)) - \left( \frac{3}{2} - \frac{3}{2} \right) \right] \]

\[ F = 20 \left[ \pi + 12 \right] \]

\[ F = 20\pi + 240 \text{ lb} \]
Math3A
Exam #03 Solutions

1. Integrate the following. Rationalize the denominator if necessary and write your solution as a single fraction.

a) \[ \int_{\pi/6}^{2\pi/3} \frac{\cot x}{\sin x} \, dx \]

b) \[ \int_{0}^{\pi/4} \sec^3 x \tan x \, dx \]
2. Evaluate the integral by making the appropriate substitutions. Write your solution as a single fraction.

\[ a) \int_{\frac{\pi}{9}}^{\frac{\pi}{2}} \frac{\sec^2 \left( \frac{3}{x} \right)}{x^2} \, dx \]

\[ b) \int_{-1}^{\frac{1}{2}} \frac{2x-5}{\sqrt{2x+3}} \, dx \]
3. Given, \( \int_{-2}^{3} f(x) \, dx = -2 \), \( \int_{-2}^{3} g(x) \, dx = -4 \), \( \frac{2}{3} \int_{-2}^{3} h(x) \, dx = 1 \), find the following:

   a) \( 2 \int_{-2}^{3} [g(x) - 2x] \, dx \)
   
   b) \( \int_{-2}^{3} \left[ 2f(x) - \frac{3}{2}h(x) \right] \, dx \)

4. A particle moves along the \( s \)-axis. Use the given information to find the position function of the particle.

   \[ a(t) = 4 - \sin 3t \quad ; \quad v\left(\frac{\pi}{6}\right) = 0 \quad ; \quad s(0) = \pi \]
5. A particle moves with acceleration $a(t) \text{ m/s}^2$ along an $s$-axis and has velocity $v_0 \text{ m/s}$ at time $t = 0$. Find the displacement and the distance traveled by the particle during the given time interval.

$$a(t) = -8t + 12; \quad v(0) = -8; \quad -2 \leq t \leq 2$$
6. A solid has an elliptical base with equation \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \). Parallel cross-sections perpendicular to the x-axis are squares. Find the volume.
7. Answer the following questions.

a) The region shown in the figure to the right is bounded by \( y = x^2, \ y = x + 2 \) and \( x = 0 \). Find the area of the bounded region.

b) The region shown in the figure to the right is bounded by \( y = x^2, \ y = x \) and \( x = 0 \). Find the volume of the solid whose base is the shaded region where parallel cross-sections perpendicular to the x-axis are semi-circles.
8. Find the average value of the function \( f(x) = \sqrt{3x} \) on the interval \([4, 9]\).

9. Divide the specified interval into \( n = 4 \) subintervals of equal length and then compute

\[
\sum_{k=1}^{4} f(x_k^*) \Delta x \text{ with } x_k^* \text{ as the left endpoint of the subinterval. Write your solution as a single fraction.}
\]

\[
f(x) = \cos x \quad \left[ 0, \frac{2\pi}{3} \right]
\]
1. Integrate the following. Rationalize the denominator if necessary and write your solution as a single fraction.

a) \( \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \cot x \ \sin x \ dx \)

\[
\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \cot x \cdot \frac{1}{\sin x} \ dx
\]

\[
\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \frac{\cos x}{\sin^2 x} \ dx
\]

\[u = \sin x\]

\[du = \cos x \ dx\]

\[x = \frac{\pi}{6} \Rightarrow u = \sin \frac{\pi}{6} = \frac{1}{2}\]

\[x = \frac{2\pi}{3} \Rightarrow u = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}\]

\[
\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{u} \ du
\]

\[
\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} u^{-2} \ du
\]

\[- \left. \frac{1}{u} \right|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}
\]

\[- \frac{2\sqrt{3}}{3} + 2\]

\[\left(- \frac{2}{\sqrt{3}}\right) - (-2)
\]

\[-\frac{2\sqrt{3}}{3} + 2\]

\[\frac{-2\sqrt{3} + 6}{3}\]

b) \( \int_{0}^{\frac{\pi}{3}} \sec^3 x \ \tan x \ dx \)

\[
\int_{0}^{\frac{\pi}{3}} \sec^2 x \cdot \sec x \ \tan x \ dx
\]

\[u = \sec x\]

\[du = \sec x \ \tan x \ dx\]

\[x = \frac{\pi}{3} \Rightarrow u = \sec \frac{\pi}{3} = 2\]

\[x = 0 \Rightarrow u = \sec 0 = 1\]

\[
\int_{1}^{2} u^2 \ du
\]

\[
\frac{u^3}{3} \bigg|_{1}^{2}
\]

\[
\frac{8}{3} - \frac{1}{3}
\]

\[
\frac{7}{3}
\]
2. Evaluate the integral by making the appropriate substitutions. Write your solution as a single fraction.

a) \[ \int \frac{18}{x^2} \sec^2 \left( \frac{2}{x} \right) \, dx \]

\[ U = \frac{3}{x} \quad \frac{d}{dx} = U = \frac{3}{x^2} \]

\[ \frac{du}{dx} = \frac{2}{x^2} \quad x = \frac{\pi}{6} \Rightarrow U = \frac{3}{\pi} \]

\[ \frac{1}{3} \int \sec^2 U \, du \]

\[ \frac{1}{3} \int \sec^2 U \, du \]

\[ \frac{1}{3} \left[ \tan U \right]_{\frac{\pi}{6}}^{\pi} \]

\[ \frac{1}{3} \left[ \tan \frac{\pi}{6} - \tan \frac{\pi}{2} \right] \]

\[ \frac{1}{3} \left[ \sqrt{3} - \frac{1}{\sqrt{3}} \right] \]

\[ \frac{1}{3} \left[ \frac{3\sqrt{3}}{3} - \frac{\sqrt{3}}{3} \right] \]

\[ \frac{1}{3} \left[ \frac{2\sqrt{3}}{3} \right] \]

\[ \frac{2\sqrt{3}}{9} \]

b) \[ \int \frac{\frac{2x}{2} - 5}{\sqrt{x/3} + 3} \, dx \]

\[ U = 2x + 3 \quad 2x + 3 = u \]

\[ du = 2 \, dx \quad u = \frac{3}{2} \Rightarrow U = 2(\frac{1}{2}) + 3 = 4 \]

\[ \frac{1}{2} \int \left[ \frac{u - 8}{u^{1/2}} \right] \, du \]

\[ \frac{1}{2} \int \left[ U^{\frac{1}{2}} - 8u^{-\frac{1}{2}} \right] \, du \]

\[ \frac{1}{2} \left[ \frac{2}{3} U^{\frac{3}{2}} - 16 \cdot 2 \right]_{1}^{4} \]

\[ \frac{1}{2} \left[ \left( \frac{2}{3} \cdot 8 - 16 \cdot 2 \right) - \left( \frac{2}{3} - 16 \right) \right] \]

\[ \frac{1}{2} \left[ \left( \frac{16}{3} - \frac{96}{3} - \frac{2}{3} + \frac{48}{3} \right) \right] \]

\[ \frac{1}{2} \left[ \frac{-34}{3} \right] \]

\[ -\frac{17}{3} \]
3. Given, \( \int_{-2}^3 f(x) \, dx = -2, \quad \int_{-2}^3 g(x) \, dx = -4, \quad \frac{2}{3} \int_{-2}^3 h(x) \, dx = 1 \), find the following:

a) \( 2 \int_{-2}^3 [g(x) - 2x] \, dx \)
\[
2 \int_{-2}^3 g(x) \, dx - 2 \int_{-2}^3 2x \, dx
\]
\[
2(-4) - 2 \left[ x^2 \right]_{-2}^3
\]
\[
-8 - 2[9 - 4]
\]
\[
-8 - 2(5)
\]
\[
-8 - 10
\]
\[
\boxed{-18}
\]

b) \( \int_{-2}^3 [2f(x) - \frac{3}{2}h(x)] \, dx \)
\[
2 \int_{-2}^3 f(x) \, dx - \frac{3}{2} \int_{-2}^3 h(x) \, dx
\]
\[
2(-4) - \frac{3}{2} \left[ \frac{33}{2} \right]
\]
\[
-8 - \frac{9}{4}
\]
\[
\frac{-16 - 9}{4}
\]
\[
\boxed{-\frac{25}{4}}
\]

4. A particle moves along the s-axis. Use the given information to find the position function of the particle.

\[ a(t) = 4 - \sin t; \quad v\left(\frac{\pi}{2}\right) = 0; \quad s(0) = \pi \]

\[ V(t) = \int a(t) \, dt \]
\[ V(t) = \int (4 - \sin t) \, dt \]
\[ V(t) = 4t + \cos t + C \]

Using \( \left(\frac{\pi}{2}, 0\right) \),
\[ 0 = 4\left(\frac{\pi}{2}\right) + \cos \left(\frac{\pi}{2}\right) + C \]
\[ 0 = 2\pi + C \]
\[ -2\pi = C \]
\[ V(t) = 4t + \cos t - 2\pi \]

\[ S(t) = \int V(t) \, dt \]
\[ S(t) = \int (4t + \cos t - 2\pi) \, dt \]
\[ S(t) = 2t^2 + \sin t - 2\pi t + C \]

Using \( (0, \pi) \),
\[ \pi = 2(0)^2 + \sin 0 - 2\pi(0) + C \]
\[ \pi = C \]
\[ S(t) = 2t^2 + \sin t - 2\pi t + \pi \]
5. A particle moves with acceleration $a(t) \, m/s^2$ along an $s$-axis and has velocity $v_0 \, m/s$ at time $t = 0$. Find the displacement and the distance traveled by the particle during the given time interval. $a(t) = -8t + 12$; $v(0) = -8$; $-2 \leq t \leq 2$

$$v(t) = \int a(t) \, dt$$
$$= \int (-8t + 12) \, dt$$
$$v(t) = -4t^2 + 12t + C$$

$\text{Ms. Ng} \quad v(0) = -8$,
$$-8 = -4(0)^2 + 12(0) + C$$
$$-8 = C$$

$$v(t) = -4t^2 + 12t - 8$$

$\text{Distance} = \int_{-2}^{2} |v(t)| \, dt$

$$= \int_{-2}^{1} (4t^2 - 12t + 8) \, dt + \int_{1}^{2} (-4t^2 + 12t - 8) \, dt$$
$$= \frac{4}{3} t^3 - 6t^2 + 8t \bigg|_{-2}^{1} + \frac{4}{3} t^3 + 6t^2 - 8t \bigg|_{1}^{2}$$
$$= \frac{4}{3} - 6 + 8 - \left( \frac{32}{3} - 24 - 16 \right) - \frac{32}{3} + 24 - 16 - \left( -\frac{4}{3} + 6 - 8 \right)$$
$$= \frac{4}{3} + 2 + \frac{32}{3} + 24 + 16 - \frac{32}{3} + 8 + \frac{4}{3} - 6 + 8$$
$$= \frac{7}{3} + 58 - 6$$
$$= \frac{8}{3} + 52$$
$$= \frac{8}{3} + \frac{156}{3}$$
$$= \frac{164}{3} \, m$$
6. A solid has an elliptical base with equation \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \). Parallel cross-sections perpendicular to the x-axis are squares. Find the volume.

\[
\begin{align*}
\frac{y^2}{9} &= 1 - \frac{x^2}{4} \\
y^2 &= 9 - \frac{9}{4}x^2 \\
y &= \sqrt{9 - \frac{9}{4}x^2} \\
y &= \frac{3}{2}\sqrt{4 - x^2}
\end{align*}
\]

\[
V = \int_{-2}^{2} 9(4-x^2) \, dx
\]

\[
= 18 \int_{0}^{2} (4-x^2) \, dx
\]

\[
= 18 \left[ 4x - \frac{x^3}{3} \right]_{0}^{2}
\]

\[
= 18 \left[ 8 - \frac{8}{3} - 0 \right]
\]

\[
= 18 \left[ \frac{24 - 8}{3} \right]
\]

\[
= \frac{18 \cdot 16}{3}
\]

\[
= 6 \cdot 16
\]

\[
= \boxed{96}
\]
7. Answer the following questions.

a) The region shown in the figure to the right is bounded by \( y = x^2, \ y = x + 2 \) and \( x = 0 \). Find the area of the bounded region.

\[
\int_{0}^{2} \left[ f(x) - g(x) \right] \, dx
\]

\[
\int_{0}^{2} \left[ x + 2 - x^2 \right] \, dx
\]

\[
\frac{x^2}{2} + 2x - \frac{x^3}{3} \bigg|_0^2
\]

\[
2 + 4 - \frac{8}{3} - 0
\]

\[
2\frac{8}{3} = \frac{16}{3}
\]

\[
\frac{18 - 8}{3} = \frac{10}{3}
\]

b) The region shown in the figure to the right is bounded by \( y = x^2, \ y = x \) and \( x = 0 \). Find the volume of the solid whose base is the shaded region where parallel cross-sections perpendicular to the \( x \)-axis are semi-circles.

\[
V = \frac{\pi}{8} \int_{0}^{1} \left( x^2 - 2x^3 + x^4 \right) \, dx
\]

\[
= \frac{\pi}{8} \left[ \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \bigg|_0^1 \right]
\]

\[
= \frac{\pi}{8} \left[ \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - 0 \right]
\]

\[
= \frac{\pi}{8} \left[ \frac{10 - 15 + 6}{30} \right]
\]

\[
= \frac{\pi}{8} \left[ \frac{1}{30} \right]
\]

\[
= \frac{\pi}{240}
\]
8. Find the average value of the function \( f(x) = 3\sqrt{x} \) on the interval \([4, 9]\).

\[
\bar{f}_{\text{avg}} = \frac{1}{9 - 4} \int_{4}^{9} 3\sqrt{x} \, dx
\]

\[
= \left. \frac{1}{9 - 4} \frac{3}{2} x^{\frac{3}{2}} \right|_{4}^{9}
\]

\[
= \frac{3}{5} \left[ 9^{\frac{3}{2}} - 4^{\frac{3}{2}} \right]
\]

\[
= \frac{3}{5} \left[ 27 - 8 \right]
\]

9. Divide the specified interval into \( n = 4 \) subintervals of equal length and then compute \( \sum_{k=1}^{4} f(x_k^*) \Delta x \) with \( x_k^* \) as the left endpoint of the subinterval. Write your solution as a single fraction.

\[
\Delta x = \frac{b - a}{n} = \frac{9 - 4}{4} = \frac{\pi}{6}
\]

\[
f(x) = \cos x \quad \left[ 0, \frac{2\pi}{3} \right]
\]

\[
\sum_{k=1}^{4} f(x_k^*) \Delta x = f(x_1^*) \Delta x + f(x_2^*) \Delta x + f(x_3^*) \Delta x + f(x_4^*) \Delta x
\]

\[
= \frac{\pi}{6} \left[ f(0) + f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{2}\right) \right]
\]

\[
= \frac{\pi}{6} \left[ \cos 0 + \cos \frac{\pi}{6} + \cos \frac{\pi}{3} + \cos \frac{\pi}{2} \right]
\]

\[
= \frac{\pi}{6} \left[ 1 + \frac{\sqrt{3}}{2} + \frac{1}{2} + 0 \right]
\]

\[
= \frac{\pi}{6} \left[ \frac{3}{2} + \frac{\sqrt{3}}{2} \right]
\]

\[
= \frac{\pi \left( \frac{3}{2} + \frac{\sqrt{3}}{2} \right)}{12}
\]