9.2 Monotone Sequences

**Definition 9.2.1**

A sequence \( \{a_n\}_{n=1}^{\infty} \) is called **strictly increasing** if:

\[
a_1 < a_2 < a_3 < \ldots < a_n < \ldots
\]

The sequence is **increasing** if:

\[
a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq \ldots
\]

"strictly decreasing" if:

\[
a_1 > a_2 > a_3 > \ldots > a_n > \ldots
\]

"decreasing" if:

\[
a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq \ldots
\]

Here is an example of a sequence that is strictly decreasing:

\[
\left\{ \frac{1}{n} \right\}_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots
\]

**Note:** This sequence has a lower bound of 0.

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0
\]

This sequence converges to 0.
An increasing sequence looks like the following:

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ...

A sequence that is either increasing or decreasing is said to be **monotone**.

If a sequence is strictly increasing or strictly decreasing, it is said to be **strictly monotone**.

**Tests for Monotonicity** (Table 9.2.2):

<table>
<thead>
<tr>
<th></th>
<th>Difference</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strictly Increasing</td>
<td>$a_{n+1} - a_n &gt; 0$</td>
<td>$\frac{a_{n+1}}{a_n} &gt; 1$</td>
</tr>
<tr>
<td>Strictly Decreasing</td>
<td>$a_{n+1} - a_n &lt; 0$</td>
<td>$\frac{a_{n+1}}{a_n} &lt; 1$</td>
</tr>
<tr>
<td>Increasing</td>
<td>$a_{n+1} - a_n \geq 0$</td>
<td>$\frac{a_{n+1}}{a_n} \geq 1$</td>
</tr>
<tr>
<td>Decreasing</td>
<td>$a_{n+1} - a_n \leq 0$</td>
<td>$\frac{a_{n+1}}{a_n} \leq 1$</td>
</tr>
</tbody>
</table>
TABLE 9.2.3 DERIVATIVE OF $f(x)$ FOR $x \geq 1$

<table>
<thead>
<tr>
<th>$f'(x)$</th>
<th>CONCLUSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&gt; 0$</td>
<td>STRICTLY INCREASING</td>
</tr>
<tr>
<td>$&lt; 0$</td>
<td>STRICTLY DECREASING</td>
</tr>
<tr>
<td>$\geq 0$</td>
<td>INCREASING</td>
</tr>
<tr>
<td>$\leq 0$</td>
<td>DECREASING</td>
</tr>
</tbody>
</table>

**DEFINITION 9.2.2**

IF DISCARDING A FINITE NUMBER OF TERMS FROM THE BEGINNING OF A SEQUENCE PRODUCES A SEQUENCE WITH A CERTAIN PROPERTY, THEN THE ORIGINAL SEQUENCE IS SAID TO HAVE THAT PROPERTY EVENTUALLY.

$\{2, 1, \frac{1}{2}, \frac{1}{3}, 1, 2, 3, 4\}$

→ EVENTUALLY STRICTLY INCREASING

**THEOREM 9.2.3**

IF A SEQUENCE $\{a_n\}_{n=1}^{\infty}$ IS EVENTUALLY INCREASING, THEN THERE ARE TWO POSSIBILITIES.

1) THERE IS AN UPPER BOUND "M" SUCH THAT $a_n \leq M$ FOR ALL $n$, IN WHICH CASE THE SEQUENCE CONVERGES TO $L \leq M$.

2) NO UPPER BOUND EXISTS.

\[\lim_{n \to \infty} a_n = +\infty ; \text{ DIVERGENT}\]
THEOREM 9.2.4
IF A SEQUENCE \( \{a_n\}_{n=1}^{\infty} \) IS EVENTUALLY DECREASING, THEN THERE ARE TWO POSSIBILITIES:
1) THERE IS A LOWER BOUND \( M \) SUCH THAT \( a_n \geq M \) FOR ALL \( n \), IN WHICH CASE THE SEQUENCE CONVERGES TO \( L \equiv M \).
2) NO LOWER BOUND EXISTS.
\[ \lim_{n \to \infty} a_n = -\infty ; \text{ DIVERGENT} \]

EX1) USE THE DIFFERENCE \( a_{n+1} - a_n \) TO SHOW THE SEQUENCE IS STRICTLY INCREASING OR STRICTLY DECREASING.
\[
\left\{ \frac{n}{2n+1} \right\}_{n=1}^{\infty}
\]
\[
a_n = \frac{n}{2n+1}
\]
\[
a_{n+1} = \frac{n+1}{2(n+1)+1}
\]
\[
= \frac{n+1}{2n+3}
\]
\[
\frac{n+1}{2n+3} - \frac{n}{2n+1} = \frac{n+1 (2n+1) - n(2n+3)}{(2n+3)(2n+1)}
\]
\[
= \frac{2n^2 + 3n + 1 - 2n^2 - 2n}{(2n+3)(2n+1)}
\]
\[
= \frac{1}{(2n+3)(2n+1)} > 0, \quad n \geq 1
\]

\[ \therefore \left\{ \frac{n}{2n+1} \right\}_{n=1}^{\infty} \text{ IS STRICTLY INCREASING} \]
EX2) USE \( \frac{a_{n+1}}{a_n} \) TO SHOW \( \left\{ \frac{n^n}{n!} \right\}_{n=1}^{\infty} \) IS STRICTLY MONOTONE.

\[
a_n = \frac{n^n}{n!}
\]

\[
a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}
\]

\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n^n}{(n+1)!} = \frac{(n+1)^n}{n!} = \left( 1 + \frac{1}{n} \right)^n > 1 \text{ FOR ALL } n \geq 1
\]

\[
\therefore \left\{ \frac{n^n}{n!} \right\}_{n=1}^{\infty} \text{ IS STRICTLY INCREASING.}
\]

EX3) DETERMINE THE MONOTONICITY OF THE SEQUENCE 
\[
\left\{ \frac{2^n}{1+2^n} \right\}_{n=1}^{\infty}
\]

HERE, WE WILL USE \( \frac{a_{n+1}}{a_n} \)

\[
a_n = \frac{2^n}{1+2^n}
\]

\[
a_{n+1} = \frac{2^{n+1}}{1+2^{n+1}}
\]

\[
\frac{a_{n+1}}{a_n} = \frac{2(1+2^n)}{1+2^{n+1}} \cdot \frac{1+2^n}{2^n} = \frac{1}{1+2^{n+1}} + 1 = 1 + \frac{1}{1+2^n} > 1 \text{ FOR ALL } n \geq 1
\]

\[
\therefore \left\{ \frac{2^n}{1+2^n} \right\}_{n=1}^{\infty} \text{ IS STRICTLY INCREASING.}
\]
EX4) DETERMINE THE MONOTONICITY OF
\[ \left\{ n e^{-2n} \right\}_{n=1}^{\infty} \]

Let \( f(y) = ye^{-2y} \quad x \geq 1 \)

\[
f'(y) = e^{-2y} + y \cdot e^{-2y} \cdot (-2)
\]

\[
= e^{-2y} - 2ye^{-2y}
\]

\[
= e^{-2y}(1 - 2y)
\]

\[
f'(x) < 0 \quad x \geq 1
\]

\[
\therefore \left\{ n e^{-2n} \right\}_{n=1}^{\infty} \text{ is strictly decreasing}
\]