Techniques of Differentiation
Selected Problems

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Techniques of Differentiation: Selected Problems

1. Find \( dy/dx \):

(a) \[ y = 4x^7 \]
\[ \frac{dy}{dx} = \frac{d}{dx}(4x^7) \]
\[ = (7)4x^6 \]
\[ = 28x^6 \]

(b) \[ y = \frac{1}{2}(x^4 + 7) \]
\[ \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}(x^4 + 7)\right) \]
\[ = \frac{1}{2}(4x^3) \]
\[ = 2x^3 \]

(c) \[ y = \pi^3 \]
\[ \frac{dy}{dx} = \frac{d}{dx}(\pi^3) \]
\[ = 0 \]

(d) \[ y = \sqrt{2}x + (1/\sqrt{2}) \]
\[ \frac{dy}{dx} = \frac{d}{dx}(\sqrt{2}x + (1/\sqrt{2})) \]
\[ = \sqrt{2} \frac{d}{dx}(x) + \frac{d}{dx}(1/\sqrt{2}) \]
\[ = \sqrt{2}(1) + 0 \]
\[ = \sqrt{2} \]
2. Find \( f'(x) \):

(a) \( f(x) = x^{-3} + \frac{1}{x^7} \)

\[
f'(x) = \frac{d}{dx}(x^{-3} + x^{-7}) = \frac{d}{dx}(x^{-3}) + \frac{d}{dx}(x^{-7}) = (-3)x^{-3-1} + (-7)x^{-7-1} = -3x^{-4} - 7x^{-8}
\]

(b) \( f(x) = \sqrt{x} + \frac{1}{x} \)

\[
f'(x) = \frac{d}{dx}(x^{1/2} + x^{-1}) = \frac{1}{2}x^{1/2-1} + (-1)x^{-1-1} = \frac{1}{2}x^{-1/2} - x^{-2} = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}
\]

(c) \( f(x) = -3x^{-8} + 2\sqrt{x} \)

\[
f'(x) = (-8)(-3)x^{-9} + \left(\frac{1}{2}\right)2x^{-1/2} = 24x^{-9} + \frac{1}{\sqrt{x}}
\]

(d) \( f(x) = ax^3 + bx^2 + cx + d \quad (a, b, c, d \text{ constant}) \)

\[
f'(x) = 3ax^2 + 2bx + c
\]
3. Find \( y'(1) \):

(a) \[ y = y(x) = 5x^2 - 3x + 1 \]
\[ y'(x) = 10x - 3 \]
\[ y'(1) = 10(1) - 3 = 7 \]

(b) \[ y = y(x) = \frac{x^{3/2} + 2}{x} \]
\[ = x^{-1}(x^{3/2} + 2) \]
\[ = x^{3/2-1} + 2x^{-1} \]
\[ = x^{1/2} + 2x^{-1} \]
\[ y'(x) = \left( \frac{1}{2} \right) x^{-1/2} - 2x^{-2} \]
\[ = \frac{1}{2\sqrt{x}} - \frac{2}{x^2} \]
\[ y'(1) = \frac{1}{2\sqrt{1}} - \frac{2}{1^2} \]
\[ = \frac{1}{2} - 2 \]
\[ = \frac{1}{2} - \frac{4}{2} = -\frac{3}{2} \]

4. Find \( dx/dt \):

(a) \[ x = t^2 - t \]
\[ \frac{dx}{dt} = \frac{d}{dt}(t^2 - t) \]
\[ = 2t - 1 \]
(b) \[
x = \frac{t^2 + 1}{3t} \\
= \frac{t^2}{3t} + \frac{1}{3t} \\
= \frac{t}{3} + \frac{1}{3t^{-1}} \\
dx \quad dt = \frac{d}{dt} \left(\frac{t}{3} + \frac{1}{3t^{-1}}\right) \\
= \frac{1}{3} + \frac{(-1)}{3}t^{-2} \\
= \frac{1}{3} - \frac{1}{3t^2}
\]

5. Find \(dy/dx\)\(x=1\) for the following:

(a) \[
y = 1 + x + x^2 + x^3 + x^4 + x^5 \\
\frac{dy}{dx} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 \\
\frac{dy}{dx}\bigg|_{x=1} = 1 + 2(1) + 3(1)^2 + 4(1)^3 + 5(1)^4 \\
= 1 + 2 + 3 + 4 + 5 = 15
\]

(b) \[
y = \frac{1 + x + x^2 + x^3 + x^4 + x^5 + x^6}{x^3} \\
= x^{-3}(1 + x + x^2 + x^3 + x^4 + x^5 + x^6) \\
= x^{-3} + x^{-2} + x^{-1} + 1 + x + x^2 + x^3 \\
\frac{dy}{dx} = -3x^{-4} - 2x^{-3} - x^{-2} + 0 + 1 + 2x + 3x^2 \\
= -\frac{3}{x^4} - \frac{2}{x^3} - \frac{1}{x^2} + 1 + 2x + 3x^2 \\
\frac{dy}{dx}\bigg|_{x=1} = -\frac{3}{1^4} - \frac{2}{1^3} - \frac{1}{1^2} + 1 + 2(1) + 3(1)^2 \\
= -3 - 2 - 1 + 1 + 2 + 3 = 0
\]
(c) \[ y = (1 - x)(1 + x)(1 + x^2)(1 + x^4) \]
\[ = (1 - x^2)(1 + x^2)(1 + x^4) \]
\[ = (1 - x^4)(1 + x^4) \]
\[ = 1 - x^8 \]
\[ \frac{dy}{dx} = -8x^7 \]
\[ \left. \frac{dy}{dx} \right|_{x=1} = -8(1)^7 = -8 \]

6. Given: \( f(x) = x^3 - 3x + 1 \), Approximate \( f'(1) \) by considering the difference quotient
\[ \frac{f(1 + h) - f(1)}{h} \]
for values of \( h \) near 0, and then find the exact value of \( f'(1) \) by differentiating.

(a) Choosing \( h = 0.01 \) we obtain the following:
\[ \frac{f(1 + 0.01) - f(1)}{0.01} = \frac{f(1.01) + f(1)}{0.01} \]
\[ = \frac{((1.01)^3 - 3(1.01) + 1) - ((1)^3 - 3(1) + 1)}{0.01} \]
\[ = \frac{1.030301 - 3.03 + 1 - 1 + 3 - 1}{0.01} \]
\[ = \frac{0.000301}{0.01} \]
\[ = 0.0301 \]
\[ \approx f'(1) \]

Note that choosing \( h \) values closer and closer to zero, i.e. \( h = 0.001, 0.0001, 0.00001, 0.000001 \ldots \), will give better and better approximations to \( f'(1) \).
(b) The exact value of \( f'(1) \):
\[
\begin{align*}
f(x) &= x^3 - 3x + 1 \\
f'(x) &= 3x^2 - 3 \\
f'(1) &= 3(1)^2 - 3 \\
&= 3 - 3 \\
&= 0
\end{align*}
\]

7. Find the indicated derivative:
(a) \[
\frac{d}{dt}[16t^2] = (2)(16t) = 32t
\]
(b) \[
\frac{dc}{dr}, \text{ where } C = 2\pi r.
\]
\[
\frac{dC}{dr} = \frac{d}{dr}(2\pi r) = 2\pi \frac{d}{dr}(r) = 2\pi
\]

8. A spherical balloon is being inflated:
(a) Find the formula for the instantaneous rate of change of the volume \( V \) with respect to the radius \( r \), given that \( V = \frac{4}{3}\pi r^3 \).
Solution: We are asked to find the change in \( V \) with respect to \( r \), in other words, find \( \frac{dV}{dr} \).
\[
\frac{dV}{dr} = (3)\frac{4}{3}\pi r^2 = \frac{4\pi r^2}{3}
\]
(b) Find the rate of change of \( V \) with respect to \( r \) at the instant when the radius is \( r = 5 \).
Solution:
\[
\frac{dV}{dr}\bigg|_{r=5} = 4\pi(5)^2 = 4\pi 25 = 100\pi
\]
9. Solve the following:

(a) Find $y'''(0)$, where $y = 4x^4 + 2x^3 + 3$

$y' = 16x^3 + 6x^2$
$y'' = 48x^2 + 12x$
$y''' = 96x + 12$
$y'''(0) = 96(0) + 12 = 12$

(b) Find $\frac{d^4y}{dx^4} \bigg|_{x=1}$, where $y = \frac{6}{x^4}$

$y = 6x^{-4}$
$\frac{dy}{dx} = (-4)6x^{-5} = -24x^{-5}$
$\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = (-5)(-24)x^{-6} = 120x^{-6}$
$\frac{d}{dx} \left( \frac{d^2y}{dx^2} \right) = \frac{d^3y}{dx^3} = (-6)120x^{-7} = -720x^{-7}$
$\frac{d}{dx} \left( \frac{d^3y}{dx^3} \right) = \frac{d^4y}{dx^4} = (-7)(-720)x^{-8} = 5040x^{-8}$
$\frac{d^4y}{dx^4} \bigg|_{x=1} = 5040(1)^{-8} = 5040$

(c) Show that $y = x^3 + 3x + 1$ satisfies $y''' + xy'' - 2y' = 0$

i. First find the derivatives needed:

$y = x^3 + 3x + 1$
$y' = 3x^2 + 3$
$y'' = 6x$
$y''' = 6$

ii. Substitute:

$y''' + xy'' - 2y' = 6 + x(6x) - 2(3x^2 + 3)$
$= 6 + 6x^2 - 6x^2 - 6$
$= 0$
10. Find a function \( y = ax^2 + bx + c \) whose graph has an \( x \)-intercept at 1, a \( y \)-intercept of -2, and a tangent line with a of slope of -1 at the \( y \)-intercept.

The \( y \)-intercept at -2 means that when \( x = 0 \), \( y = -2 \), i.e. \((0, -2)\) is on the graph. \((This \hspace{1mm} is \hspace{1mm} the \hspace{1mm} easiest \hspace{1mm} way \hspace{1mm} to \hspace{1mm} start \hspace{1mm} because \hspace{1mm} plugging \hspace{1mm} in \hspace{1mm} x=0 \hspace{1mm} reduces \hspace{1mm} the \hspace{1mm} equation \hspace{1mm} to \hspace{1mm} find \hspace{1mm} c \hspace{1mm} immediately)\)

\[
\begin{align*}
y &= ax^2 + bx + c \\
-2 &= a(0)^2 + b(0) + c \\
-2 &= c 
\end{align*}
\]

So \( y = ax^2 + bx - 2 \). Now we use the \( x \)-intercept, which tells us that \((1, 0)\) is also on the graph, therefore,

\[
\begin{align*}
0 &= a(1)^2 + b(1) - 2 \\
0 &= a + b - 2 \\
2 &= a + b 
\end{align*}
\]

Now we also want a tangent line with a slope of -1 at the \( y \)-intercept. This means that \( \frac{dy}{dx} \bigg|_{x=0} = -1 \).

\[
\begin{align*}
y &= ax^2 + bx - 2 \\
\frac{dy}{dx} &= 2ax + b \\
\bigg|_{x=0} \frac{dy}{dx} &= -1 = 2a(0) + b \\
-1 &= b 
\end{align*}
\]

Substitute \( b = -1 \) into \( 2 = a + b \) to get \( a = 3 \). Our function is thus,

\[
y = 3x^2 - x - 2 
\]
11. Find \( k \) if the curve (i) \( y = x^2 + k \) is tangent to the line (ii) \( y = 2x \).

Let \( P \) be the point at \( (x_0, y_0) \) at which (i) and (ii) are tangent to each other. Find \( \frac{dy}{dx} \) of (i):

\[
y = x^2 + k
\]

\[
\frac{dy}{dx} = 2x \quad \text{(iii)}
\]

We need the to find \( x_0 \) so that the slopes of (iii) and (ii) are equal at \( (x_0, y_0) \). The slope of the (ii) is clearly 2:

\[2x_0 = 2\]

\[x_0 = 1\]

Now \( P \) is on the line \( y = 2x \), we can sub \( x_0 = 1 \) and obtain \( y_0 = 2 \). So \( P = (1, 2) \). Plug \( P \) into (i):

\[
y = x^2 + k
\]

\[2 = 1^2 + k\]

\[1 = k\]

12. Find the \( x \)-coordinate of the point on the graph \( y = x^2 \) where the tangent line is parallel to the secant line that cuts the curve at \( x = -1 \) and \( x = 2 \).

(Try drawing a picture to help visualize this problem.)

The secant line is the line that goes through the points \((-1, 1)\) and \((2, 4)\), (why?). The slope is then found by using the slope formula of two points:

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{2 - (-1)} = \frac{3}{3} = 1.
\]

We want this to be the same slope as the tangent line of \( y = x^2 \) which is \( \frac{dy}{dx} = 2x \). Thus \( 2x = 1 \rightarrow x = \frac{1}{2} \) is the \( x \)-coordinate we are looking for.
13. Show that any two tangent lines to the parabola $y = ax^2$, $a \neq 0$, intersect at a point that is on the vertical line halfway between the points of tangency.

$$y = ax^2$$
$$\frac{dy}{dx} = 2ax$$

Let $P_1 = (x_1, y_1 = ax_1^2)$ and $P_2 = (x_2, y_2 = ax_2^2)$ be two different points of tangency. The tangent line at $P_1$ has the equation:

$$y - y_1 = m(x - x_1)$$
$$y - ax_1^2 = \left. \frac{dy}{dx} \right|_{x=x_1} (x - x_1)$$
$$y - ax_1^2 = 2ax_1(x - x_1)$$
$$y - ax_1^2 = (2ax)x_1 - 2ax_1^2$$
$$y = (2ax)x_1 - ax_1^2$$

Similarly, the tangent line at $P_2$ is $y = (2ax)x_2 - ax_2^2$.

Set $y = y$ and solve for $x$:

$$(2ax)x_1 - ax_1^2 = (2ax)x_2 - ax_2^2$$
$$(2ax)(x_1 - x_2) = a(x_1^2 - x_2^2)$$
$$2x(x_1 - x_2) = (x_1 - x_2)(x_1 + x_2)$$
$$2x = (x_1 + x_2)$$
$$x = \frac{1}{2}(x_1 + x_2)$$

This is the $x$-coordinate of a point on the vertical line halfway between $P_1$ and $P - 2$. 
14. Show that the segment of the tangent line to the graph of \( y = \frac{1}{x}, \ x > 0 \), and the coordinate axes has an area of 2 square units.

Let \( P_0 = (x_0, y_0) \) be a point on the curve. Then \( y_0 = \frac{1}{x_0} \) and \( \frac{dy}{dx} = -\frac{1}{x^2} \).

The equation for the tangent line is:

\[
y - y_0 = m(x - x_0) \\
y - \frac{1}{x_0} = \frac{dy}{dx}|_{x=x_0}(x - x_0) \\
y - \frac{1}{x_0} = -\frac{1}{x_0^2}(x - x_0) \\
y - \frac{1}{x_0} = -\frac{x}{x_0^2} + \frac{1}{x_0} \\
y = -\frac{x}{x_0^2} + \frac{2}{x_0}
\]

This line will intersect the \( x \)-axis when \( y = 0 \).

\[
0 = -\frac{x}{x_0^2} + \frac{2}{x_0} \\
\frac{x}{x_0^2} = \frac{2}{x_0} \\
x = 2x_0
\]

The line will intersect the \( y \)-axis when \( x = 0 \).

\[
y = -\frac{0}{x_0^2} + \frac{2}{x_0} = \frac{2}{x_0}
\]

So we have a triangle with coordinates: \( (0, \frac{2}{x_0}) \), \( (0, 0) \), \( (2x_0, 0) \). The area of this triangle is: \( \frac{1}{2} \left( \frac{2}{x_0} \right) (2x_0) = 2 \).
15. Find conditions on $a$, $b$, $c$, $d$, so that the graph of the polynomial $f(x) = ax^3 + bx^2 + cx + d$ has exactly one horizontal tangent.

$f(x)$ will have a horizontal tangent when $f'(x) = 3ax^2 + 2bx + c = 0$. We can solve this by using the quadratic formula:

$$x = \frac{-2b \pm \sqrt{(2b)^2 - 4(3a)c}}{2(3a)}$$
$$= \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a}$$
$$= \frac{-2b \pm \sqrt{4(b^2 - 3ac)}}{6a}$$
$$= \frac{-2b \pm 2\sqrt{b^2 - 3ac}}{6a}$$
$$= \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

Now $x$ will have one real solution if $\sqrt{b^2 - 3ac} = 0$:

$$\left(\sqrt{b^2 - 3ac}\right)^2 = 0^2$$
$$b^2 - 3ac = 0$$
$$b^2 = 3ac$$
$$b = \pm \sqrt{3ac}$$

16. Let $f(x) = \begin{cases} x^2 - 16x, & x < 9 \\ \sqrt{x}, & x \geq 9 \end{cases}$

Is $f$ continuous at $x = 9$? Determine whether $f$ is differentiable at $x = 9$. If so, find the value of the derivative there.

$$\lim_{x \to 9^-} f(x) = (9)^2 - 16(9) = 81 - 144 = -63$$
$$\lim_{x \to 9^+} f(x) = \sqrt{9} = 3$$

Since $f(x)$ is not continuous at $x = 9$, it is not differentiable at $x = 9$.

(Differentiable implies continuous, therefore not continuous implies not differentiable.)
17. Let \[ f(x) = \begin{cases} x^3 + \frac{1}{16}, & x < 1/2 \\ \frac{3}{4}x^2, & x \geq 1/2 \end{cases} \]

Determine whether \( f \) is differentiable at \( x = \frac{1}{2} \). If so, find the value of the derivative there.

\[
\lim_{x \to \frac{1}{2}^-} f(x) = \left( \frac{1}{2} \right)^3 + \frac{1}{16} = \frac{1}{8} + \frac{1}{16} = \frac{3}{16}
\]

\[
\lim_{x \to \frac{1}{2}^+} f(x) = \frac{3}{4} \left( \frac{1}{2} \right)^2 = \frac{3}{4} \left( \frac{1}{4} \right) = \frac{3}{16}
\]

Since these are equal, \( f \) has been shown to be continuous at \( x = \frac{1}{2} \). (If \( f \) was not continuous then we would immediately be able to say that \( f \) is not differentiable) Now \( f'(x) \) is given by:

\[
f'(x) = \begin{cases} 3x^2, & x < 1/2 \\ \frac{3}{2}x, & x \geq 1/2 \end{cases}
\]

Now we must still show differentiability at \( x = \frac{1}{2} \):

\[
\lim_{x \to \frac{1}{2}^-} f'(x) = 3 \left( \frac{1}{2} \right)^2 = \frac{3}{4}
\]

\[
\lim_{x \to \frac{1}{2}^+} f'(x) = \frac{3}{2} \left( \frac{1}{2} \right) = \frac{3}{4}
\]

Therefore \( f \) is both continuous and differentiable at \( x = \frac{1}{2} \) with \( f' \left( \frac{1}{2} \right) = \frac{3}{4} \).