Rectilinear Motion Using Integration
Solutions To Selected Problems
Calculus 9th Edition Anton, Bivens, Davis

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1. A particle moves along an s-axis. Use the given information to find the position function of the particle.

(a) \( v(t) = 3t^2 - 2t \); \( s(0) = 1 \)

\[
s(t) = \int v(t) \, dt = \int 3t^2 - 2t \, dt = t^3 - t^2 + C
\]

\[s(0) = 0^3 - 0^2 + C = 1\]

\[C = 1\]

\[s(t) = t^3 - t^2 + 1\]

(b) \( a(t) = 3 \sin(3t) \); \( v(0) = 3 \); \( s(0) = 3 \)

\[
v(t) = \int a(t) \, dt = \int 3 \sin(3t) \, dt = -\cos(3t) + C
\]

\[v(0) = -\cos(0) + C = 3\]

\[-1 + C = 3\]

\[C = 4\]

\[v(t) = -\cos(4t) + 4\]

\[
s(t) = \int v(t) \, dt = \int (-\cos(3t) + 4) \, dt = -\frac{1}{3} \sin(3t) + 4t + C
\]

\[s(0) = -\frac{1}{3} \sin(0) + 4(0) + C = 3\]

\[C = 3\]

\[s(t) = -\frac{1}{3} \sin(3t) + 4t + 3\]
(c) \( a(t) = 2t^{-3}; \quad v(1) = 0; \quad s(1) = 0 \)

\[
v(t) = \int 2t^{-3} \, dt = \frac{2t^{-2}}{-2} + C = -t^{-2} + C
\]

\[
v(1) = -(-1)^{-2} + C = 0
\]
\[-1 + C = 0
\]
\[C = 1
\]
\[v(t) = -t^{-2} + 1
\]

\[
s(t) = \int (-t^{-2} + 1) \, dt = \frac{-t^{-1}}{-1} + t + C = t^{-1} + t + C
\]

\[
s(1) = 1^{-1} + 1 + C = 2
\]
\[2 + C = 2
\]
\[C = 0
\]

\[
\boxed{s(t) = t^{-1} + t}
\]

2. A particle moves with a velocity of \( v(t) \) m/s along an s-axis. Find the displacement and the distance traveled by the particle during the given time interval.

(a) \( v(t) = \sin(t); \quad 0 \leq t \leq \frac{\pi}{2} \)

Total displacement over the time interval is given by:

\[
\int_0^{\pi/2} v(t) \, dt = \int_0^{\pi/2} \sin(t) \, dt
\]
\[= -\cos(t) \bigg|_0^{\pi/2}
\]
\[= -(\cos(\pi/2) - \cos(0)) = -(0 - 1) = 1
\]

Note that \( \sin(t) \geq 0 \) over this interval so that \( |v(t)| = |\sin(t)| = \sin(t) \).

Thus, the distance traveled is still \( \int_0^{\pi/2} \sin(t) \, dt = 1 \).
(b) \( v(t) = \cos(t); \quad \frac{\pi}{2} \leq t \leq 2\pi \)

Total displacement is given by:

\[
\int_{\pi/2}^{2\pi} \cos(t) \, dt = \sin(t)|_{\pi/2}^{2\pi} \\
= \sin(2\pi) - \sin(\pi/2) = 0 - 1 = -1
\]

Now \( |v(t)| = |\cos(t)| \) is both positive and negative over this interval. (Draw a graph of \( \cos(t) \) to help see this).

\[
|\cos(t)| = \begin{cases} 
- \cos(t) & \frac{\pi}{2} \leq t \leq \frac{3\pi}{2} \\
\cos(t) & \frac{3\pi}{2} \leq t \leq 2\pi
\end{cases}
\]

So the total distance traveled is given by:

\[
\int_{\pi/2}^{2\pi} |\cos(t)| \, dt = \int_{\pi/2}^{3\pi/2} (-\cos(t)) \, dt + \int_{3\pi/2}^{2\pi} \cos(t) \, dt \\
= -\sin(t)|^{3\pi/2}_{\pi/2} + \sin(t)|^{2\pi}_{3\pi/2} \\
= -\left( \sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) \right) + \left( \sin(2\pi) - \sin\left(\frac{3\pi}{2}\right) \right) \\
= -(-1 - 1) + (0 - (-1)) \\
= -(-2) + 1 = 2 + 1 = 3
\]
3. A particle moves along with acceleration $a(t) \, m/s^2$ along an s-axis and has velocity $v_0 \, m/s$ at time $t = 0$. Find the displacement and the distance traveled by the particle during the given time interval.

$$a(t) = 3; \quad v_0 = -1; \quad 0 \leq t \leq 2$$

Note that this is constant acceleration, so we automatically know that

$$v(t) = -1 + 3t$$

Now the total displacement is given by:

$$\int_{0}^{2} (-1 + 3t) \, dt = \left[ -t + \frac{3}{2}t^2 \right]_0^2$$

$$= \left( -2 + \frac{3}{2}(2)^2 \right) - \left( -0 + \frac{3}{2}0^2 \right)$$

$$= (-2 + 3(2)) - 0 = -2 + 6 = 4$$

Now we must analyze $|v(t)| = | -1 + 3t |$, where $-1 + 3t$ is a line which is zero at $t = \frac{1}{3}$.
\[ | -1 + 3t | = \begin{cases} 
-(-1 + 3t) = 1 - 3t & 0 \leq t \leq \frac{1}{3} \\
-1 + 3t & \frac{1}{3} \leq t \leq 2 
\end{cases} \]

So the Total distance traveled is given by:

\[
\int_{0}^{2} | -1 + 3t | \, dt = \int_{0}^{1/3} (1 - 3t) \, dt + \int_{1/3}^{2} (-1 + 3t) \, dt \\
= \left( t - \frac{3}{2} t^{2} \bigg|_{0}^{1/3} \right) + \left( -t + \frac{3}{2} t^{2} \bigg|_{1/3}^{2} \right) \\
= \left( \frac{1}{3} - \frac{3}{2} \left( \frac{1}{3} \right)^{2} \right) + \left( -2 + \frac{3}{2} (2)^{2} \right) - \left( -\frac{1}{3} + \frac{3}{2} \left( \frac{1}{3} \right)^{2} \right) \\
= \frac{1}{3} - \frac{3}{2} \cdot \frac{1}{9} - 2 + \frac{1}{3} - \frac{1}{6} \\
= \frac{2}{3} - \frac{2}{6} + 4 \\
= \frac{1}{3} + 4 = \frac{1 + 12}{3} = \frac{13}{3} \]
4. In each part, use the given information to find the position, velocity, speed, and acceleration at time \( t = 1 \).

(a) \( v(t) = \sin(\pi t/2) \) \( \quad s = 0 \). when \( t = 0 \)

\[
v(1) = \sin(\pi/2) = 1
\]

\[v(1) = 1\]

\[a(t) = v'(t) = \frac{\pi}{2} \cos(\pi t/2)\]

\[a(1) = \frac{\pi}{2} \cos(\pi/2) = \frac{\pi}{2} (0) = 0\]

\[a(1) = 0\]

\[s(t) = \int \sin(\pi t/2) \, dt = -\frac{2}{\pi} \cos(\pi t/2) + C\]

\[s(0) = -\frac{2}{\pi} \cos(0) + C = 0\]

\[C = \frac{2}{\pi}\]

\[s(t) = -\frac{2}{\pi} \cos(\pi t/2) + \frac{2}{\pi}\]

\[s(1) = -\frac{2}{\pi} \cos(\pi/2) + \frac{2}{\pi} = 0 + \frac{2}{\pi} = \frac{2}{\pi}\]

\[s(1) = \frac{2}{\pi}\]
5. Suppose at time $t = 0$ a particle is at the origin of an $x$-axis and has a velocity of $v_0 = 25 \text{ cm/s}$. For the first 4 seconds thereafter it has no acceleration, and then it is acted on by a retarding force that produces a constant negative acceleration of $a = -10 \text{ cm/s}^2$.

(a) Sketch the acceleration versus time curve over the interval $0 \leq t \leq 12$.

(b) Sketch the velocity versus time curve over the time interval $0 \leq t \leq 12$. 
To determine the line $-10t + 65$ in part (b) above, we know that the slope at $t = 4$ is -10 as the acceleration there is -10. We also need the point at $t = 4$. Since the velocity is always 25 for the first four seconds, we have the point $(4,25)$. Using the point-slope formula we get:

\[ y - 25 = -10(x - 4) \]
\[ y = -10x + 40 + 25 \]
\[ y = -10x + 65 \quad for \ t \geq 4 \]

(c) Find the $x$-coordinate of the particle at times $t = 8$ seconds and $t = 12$.

First find the function $s(t) = \int v(t) \, dt$, where $v(t) = \begin{cases} 25 & 0 \leq t \leq 4 \\ -10t + 25 & 4 \leq t \leq 12 \end{cases}$

\[ s(t) = \int 25 \, dt = 25t + C \]
\[ s(0) = 25(0) + C = 0 \]
\[ C = 0 \]
\[ s(t) = 25t \quad 0 \leq t \leq 4 \]

\[ s(t) \int 65 - 10t \, dt = 65t - 5t^2 + C \]
\[ s(4) = 100 \quad from \ the \ above \ formula, \ s(t) = 25t \]
\[ 65(4) - 5(4)^2 + C = 100 \]
\[ 260 - 5(16) + C = 100 \]
\[ 180 + C = 100 \]
\[ C = -80 \]
\[ s(t) = 64t - 5t^2 - 80 \quad 4 \leq t \leq 12 \]
Now we can find $s(8)$ and $s(12)$:

$$s(8) = 65(8) - 5(8)^2 - 80$$
$$= 520 - 320 - 80 = 200 - 80 = 120$$

$s(8) = 120$

$$s(12) = 65(12) - 5(12)^2 - 80$$
$$= 780 - 720 - 80 = 60 - 80 = -20$$

$s(12) = -20$

(d) What is the maximum $x$-coordinate of the particle over the time interval $0 \leq t \leq 12$? This occurs when

$$s'(t) = v(t) = 0$$
$$\rightarrow 65 - 10t = 0$$
$$10t = 65$$
$$t = 65/10 = 6.5$$
6. Spotting a police car, you hit the brakes on your new Porsche to reduce your speed to from 90 mi/h to 60 mi/h at a constant rate over a distance of 200 feet. (Note that 88 ft/sec = 60 mi/h. So 1 mi/h = 22/15 ft/sec).

(a) Find the acceleration in $ft/s^2$.

We need to first find the $v(t)$ and $s(t)$ functions.

$$a(t) = a_0 \quad (constant)$$

$$v(t) = \int a_0 \, dt = a_0 t + v_0$$

$$v_0 = \frac{90 \text{ mi}}{\text{h}} = 90 \left( \frac{22 \text{ ft}}{15 \text{ sec}} \right) = 6(22) \frac{\text{ft}}{\text{sec}} = 132 \frac{\text{ft}}{\text{sec}}$$

$$v(t) = a_0 t + 132$$

$$s(t) = \int v(t) \, dt = \int a_0 t + 132 \, dt$$

$$= a_0 \frac{t^2}{2} + 132t + s_0$$

$$s_0 = 0 \quad (given)$$

$$s(t) = \frac{1}{2} a_0 t^2 + 132t$$

Now we use the fact that at $s = 200$, we have

$$v(t) = 60 \text{ mi/h} = 88 \text{ ft/sec}$$

$$\rightarrow 88 = a_0 t + 132$$

$$-44 = a_0 t$$

$$-\frac{44}{a_0} = t$$
Now substitute this value of $t$ into the equation $s(t) = 200 = \frac{1}{2}a_0t^2 + 132$

\[
200 = \frac{1}{2}a_0 \left( -\frac{44}{a_0} \right)^2 + 132 \left( -\frac{44}{a_0} \right)
\]
\[
200 = \frac{a_0}{2} \left( \frac{1936}{a_0^2} \right) - \frac{5808}{a_0}
\]
\[
200 = \frac{968}{a_0} - \frac{5808}{a_0}
\]
\[
200 = -\frac{4840}{a_0}
\]
\[
200a_0 = -4840
\]
\[
a_0 = -\frac{4840}{200} = -\frac{121}{5}
\]

\[
a(t) = a_0 = -\frac{121}{5}
\]

\[
\rightarrow v(t) = -\frac{121}{5}t + 132
\]
\[
\rightarrow s(t) = -\frac{121}{5} t^2 + 132t = -\frac{121}{10}t^2 + 132t = -12.1t^2 + 132t
\]

(b) How long does it take for you to reduce your speed to 55 mi/h?

Since our equations are in ft/sec, convert $55 \frac{mi}{h} = 55 \left( \frac{22 \text{ ft}}{15 \text{ sec}} \right) = \frac{242 \text{ ft}}{3 \text{ sec}}$.

Now solve using the velocity function:

\[
\frac{242}{3} = -\frac{121}{5} t + 132
\]
\[
\frac{242}{3} - 132 = -\frac{121}{5} t
\]
\[
\frac{242 - 396}{3} = -\frac{121}{5} t
\]
\[
\left( -\frac{154}{3} \right) \left( -\frac{5}{121} \right) = t
\]
\[
\frac{(2)(7)(11)(5)}{(3)(11)(11)} = t
\]
\[
t = \frac{70}{33}
\]
(c) At the acceleration obtained in part (a), how long would it take for you to bring your Porsche to a complete stop from 90 mi/h?

We are asked to find at what time $t$ is $v(t) = 0$?

\[-\frac{121}{5}t + 132 = 0\]
\[-\frac{121}{5}t = -132\]
\[t = 132 \cdot \frac{5}{121}\]
\[t = \frac{(11)(12)(5)}{(11)(11)}\]
\[t = \frac{60}{11} \text{ sec} \approx 5.45 \text{ sec}\]

7. A projectile is launched vertically upward from ground level with an initial velocity of $v_0 = 112 \text{ ft/s}$.

(a) Find the velocity at $t = 3$ and $t = 5$.

Take gravity to be the acceleration $a = -32 \text{ ft/s}^2$ so we have

$v(t) = v_0 + at = 112 - 32t$
$v(3) = 112 - 32(3) = 112 - 96 = 16 \text{ ft/s}$
$v(5) = 112 - 32(5) = 112 - 160 = -48 \text{ ft/s}$

(b) How high will the projectile rise?

Solve $v(t) = 0$ for $t$ to find the time when the particle "stops" rising to turn around and fall back. This will the time at which the particle is at its highest.
\[ 112 - 32t = 0 \]
\[ 112 = 32t \]
\[ t = \frac{112}{32} = \frac{7}{2} \]

Now we need the actual position function \( s(t) \) and then find \( s(7/2) \):

\[
s(t) = \int v(t) \, dt = \int 112 - 32t \, dt = 112t - 16t^2 + s_0 \quad (s_0 = 0)
\]
\[
s(t) = 112t - 16t^2
\]

\[
s(7/2) = 112 \cdot \frac{7}{2} - 16 \left( \frac{7}{2} \right)^2
\]
\[
= 56(7) - 16 \cdot \frac{49}{4}
\]
\[
= 392 - 196 = 196 \text{ ft}
\]

(c) Find the speed of the projectile when it hits the ground.

Find when \( s(t) = 0 \) for \( t \) to find out the time when it hits the ground. Then find \( |v(t)| \) at this time to find its speed upon impact.

\[
-16t^2 + 112t = 0
\]
\[-t(16t - 112) = 0 \]
\[ t = 0 \quad \text{Initial starting time, disregard} \]
\[
6t - 112 = 0
\]
\[ 16t = 112 \]
\[ t = 112/16 = 7 \]

\[
v(7) = -32(7) + 112 = -224 + 112 = -112
\]

So the speed upon impact is \( 112 \text{ ft/s} \).