Section 1.4: Using the Definitions of the Trigonometric Functions

I. Reciprocal Identities

The six major trigonometric functions of an angle θ are **sine**, **cosine**, **tangent**, **cosecant**, **secant**, and **tangent**. Some of these are very closely related to each other with some useful identities.

Example 1: Explore the connection between the **sine** function and the **cosecant** function.

Reciprocal Identities

For all angles θ for which both functions are defined,

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Example 2(Using the Reciprocal Identities): Find each function value.

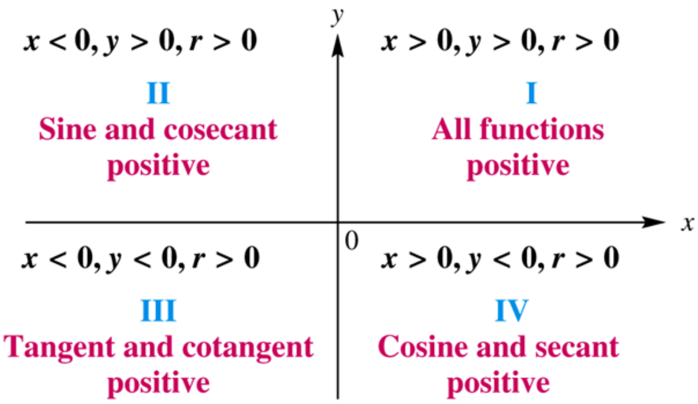
(a) tan θ, given that cot θ = 4.

(b) sec θ, given that $cos\theta = -\frac{2}{\sqrt{20}}$

II. Signs of Function Values

In our definitions of the six major functions, we assumed that r, the distance from the point (x, y) to the origin, was positive ($r > 0$). This makes the signs of each function very predictable based on which Quadrant the point (x, y) is in.

θ in Quadrant	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−



Example 3 (Determining Signs): Determine the signs of the trigonometric functions of an angle in standard position with the given measure.

(a) 54°

(b) 260°

(c) –60°

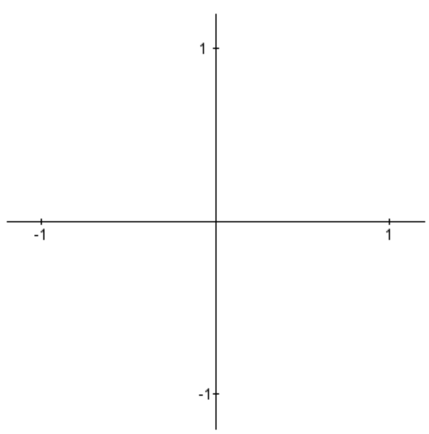
Example 4 (Identifying the Quadrant of an Angle): Identify the quadrant (or possible quadrants) of an angle θ that satisfies the given conditions.

(a) $\tan\theta > 0$, $\csc\theta < 0$

(b) $\sin\theta > 0$, $\csc\theta > 0$

III. Ranges of Function Values

- Let's explore the **range** of possible values that a **sine** or **cosine** function could take on. We will do this by focusing on points (x, y) that are on a circle of radius 1 $(r = 1)$. This circle is known as **the Unit Circle**.



Summary Table

Trigonometric Function of θ	Range (Set-Builder Notation)	Range (Interval Notation)
$\sin\theta$, $\cos\theta$	$\{y \mid \lvert y\rvert \leq 1\}$	$[-1, 1]$
$\tan\theta$, $\cot\theta$	$\{y \mid y \text{ is a real number}\}$	$(-\infty, \infty)$
$\sec\theta$, $\csc\theta$	$\{y \mid \lvert y\rvert \geq 1\}$	$(-\infty, -1] \cup [1, \infty)$

Example 5 (Range of Values): Decide whether each statement is *possible* or *impossible.*

(a) $\cot\theta = -0.999$

(b) $\cos\theta = -1.7$

(c) $\csc\theta = 0$

- The six trigonometric functions are all defined in terms of x, y, and r, and related by the **Pythagorean theorem**

$$r^2 = x^2 + y^2$$

By knowing the value of one function and the quadrant in which the angle terminates in, we can easily find the value of the other five functions.

Example 6 (Finding all Values Given Just One): Suppose the angle θ lies in quadrant III, and $\tan\theta = \frac{8}{5}$. Find the values of the other five trigonometric functions.

IV. Pythagorean Identities

- **An identity** is an equation that is true for every number that is a meaningful replacement for the variable.

  Let's derive three new and amazing identities using the relationship $r^2 = x^2 + y^2$!

Pythagorean Identities

For all angles θ for which the function values are defined, the following identities hold.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

V. Quotient Identities

- For all angles θ for which the denominators are not zero, the following identities hold:

Example 7 (Using Identities): Find cos θ and tan θ given that $\sin\theta = -\frac{\sqrt{2}}{3}$ and cos $\theta > 0$.