Techniques of Differentiation
Selected Problems

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Techniques of Differentiation: Selected Problems

1. Find dy/dx:

(a)
$$\begin{aligned} y &= 4x^7 \\ \frac{dy}{dx} &= \frac{d}{dx}(4x^7) \\ &= (7)4x^6 \\ &= \boxed{28x^6} \end{aligned}$$

(b)
$$\begin{aligned} y &= \frac{1}{2}(x^4+7) \\ \frac{dy}{dx} &= \frac{d}{dx}(\frac{1}{2}(x^4+7)) \\ &= \frac{1}{2}(4x^3) \\ &= \boxed{2x^3} \end{aligned}$$

(c)
$$\begin{aligned} y &= \pi^3 \\ \frac{dy}{dx} &= \frac{d}{dx}(\pi^3) \\ &= \boxed{0} \end{aligned}$$

(d)
$$\begin{aligned} y &= \sqrt{2}x + (1/\sqrt{2}) \\ \frac{dy}{dx} &= \frac{d}{dx}(\sqrt{2}x + (1/\sqrt{2})) \\ &= \sqrt{2}\frac{d}{dx}(x) + \frac{d}{dx}(1/\sqrt{2}) \\ &= \sqrt{2}(1) + 0 \\ &= \boxed{\sqrt{2}} \end{aligned}$$

2. <u>Find $f'(x)$:</u>

(a)
$$\begin{aligned} f(x) &= x^{-3} + \frac{1}{x^7} \\ f'(x) &= \frac{d}{dx}(x^{-3} + x^{-7}) \\ &= \frac{d}{dx}(x^{-3}) + \frac{d}{dx}(x^{-7}) \\ &= (-3)x^{-3-1} + (-7)x^{-7-1} \\ &= \boxed{-3x^{-4} - 7x^{-8}} \end{aligned}$$

(b)
$$\begin{aligned} f(x) &= \sqrt{x} + \frac{1}{x} \\ f'(x) &= \frac{d}{dx}(x^{1/2} + x^{-1}) \\ &= \frac{1}{2}x^{1/2-1} + (-1)x^{-1-1} \\ &= \frac{1}{2}x^{-1/2} - x^{-2} \\ &= \boxed{\tfrac{1}{2\sqrt{x}} - \tfrac{1}{x^2}} \end{aligned}$$

(c)
$$\begin{aligned} f(x) &= -3x^{-8} + 2\sqrt{x} \\ f'(x) &= (-8)(-3)x^{-9} + \left(\frac{1}{2}\right) 2x^{-1/2} \\ &= \boxed{24x^{-9} + \tfrac{1}{\sqrt{x}}} \end{aligned}$$

(d)
$$\begin{aligned} f(x) &= ax^3 + bx^2 + cx + d \quad (a, b, c, d \ constant) \\ f'(x) &= \boxed{3ax^2 + 2bx + c} \end{aligned}$$

3. <u>Find $y'(1)$:</u>

(a)

$$y = y(x) = 5x^2 - 3x + 1$$
$$y'(x) = 10x - 3$$
$$y'(1) = 10(1) - 3 = \boxed{7}$$

(b)

$$y = y(x) = \frac{x^{3/2} + 2}{x}$$
$$= x^{-1}(x^{3/2} + 2)$$
$$= x^{3/2-1} + 2x^{-1}$$
$$= x^{1/2} + 2x^{-1}$$

$$y'(x) = \left(\frac{1}{2}\right) x^{-1/2} - 2x^{-2}$$
$$= \frac{1}{2\sqrt{x}} - \frac{2}{x^2}$$

$$y'(1) = \frac{1}{2\sqrt{1}} - \frac{2}{1^2}$$
$$= \frac{1}{2} - 2$$
$$= \frac{1}{2} - \frac{4}{2} = \boxed{-\tfrac{3}{2}}$$

4. <u>Find dx/dt:</u>

(a)

$$x = t^2 - t$$
$$\frac{dx}{dt} = \frac{d}{dt}(t^2 - t)$$
$$= \boxed{2t - 1}$$

(b)

$$\begin{aligned}
x &= \frac{t^2+1}{3t} \\
&= \frac{t^2}{3t} + \frac{1}{3t} \\
&= \frac{1}{3}t + \frac{1}{3}t^{-1} \\
\frac{dx}{dt} &= \frac{d}{dt}(\frac{1}{3}t + \frac{1}{3}t^{-1}) \\
&= \frac{1}{3} + (-1)\frac{1}{3}t^{-2} \\
&= \boxed{\tfrac{1}{3} - \tfrac{1}{3t^2}}
\end{aligned}$$

5. <u>Find $dy/dx|_{x=1}$ for the following:</u>

(a)

$$\begin{aligned}
y &= 1 + x + x^2 + x^3 + x^4 + x^5 \\
\frac{dy}{dx} &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 \\
\left.\frac{dy}{dx}\right|_{x=1} &= 1 + 2(1) + 3(1)^2 + 4(1)^3 + 5(1)^4 \\
&= 1 + 2 + 3 + 4 + 5 = \boxed{15}
\end{aligned}$$

(b)

$$\begin{aligned}
y &= \frac{1 + x + x^2 + x^3 + x^4 + x^5 + x^6}{x^3} \\
&= x^{-3}(1 + x + x^2 + x^3 + x^4 + x^5 + x^6) \\
&= x^{-3} + x^{-2} + x^{-1} + 1 + x + x^2 + x^3 \\
\frac{dy}{dx} &= -3x^{-4} - 2x^-3 - x^{-2} + 0 + 1 + 2x + 3x^2 \\
&= -\frac{3}{x^4} - \frac{2}{x^3} - \frac{1}{x^2} + 1 + 2x + 3x^2 \\
\left.\frac{dy}{dx}\right|_{x=1} &= -\frac{3}{1^4} - \frac{2}{1^3} - \frac{1}{1^2} + 1 + 2(1) + 3(1)^2 \\
&= -3 - 2 - 1 + 1 + 2 + 3 = \boxed{0}
\end{aligned}$$

(c)

$$\begin{aligned} y &= (1-x)(1+x)(1+x^2)(1+x^4) \\ &= (1-x^2)(1+x^2)(1+x^4) \\ &= (1-x^4)(1+x^4) \\ &= 1-x^8 \\ \frac{dy}{dx} &= -8x^7 \\ \left.\frac{dy}{dx}\right|_{x=1} &= -8(1)^7 = \boxed{\text{-8}} \end{aligned}$$

6. *Given* : $f(x) = x^3 - 3x + 1$, Approximate $f'(1)$ by considering the difference quotient

$$\frac{f(1+h) - f(1)}{h}$$

for values of h near 0, and then find the exact value of $f'(1)$ by differentiating.

(a) Choosing $h = 0.01$ we obtain the following:

$$\begin{aligned} \frac{f(1+0.01) - f(1)}{0.01} &= \frac{f(1.01) + f(1)}{0.01} \\ &= \frac{((1.01)^3 - 3(1.01) + 1) - ((1)^3 - 3(1) + 1)}{0.01} \\ &= \frac{1.030301 - 3.03 + 1 - 1 + 3 - 1}{0.01} \\ &= \frac{0.000301}{0.01} \\ &= 0.0301 \\ &\approx f'(1) \end{aligned}$$

Note that choosing h values closer and closer to zero, i.e. $h = 0.001, 0.0001, 0.00001,$ $0.000001\ldots$, will give better and better approximations to $f'(1)$.

(b) The exact value of $f'(1)$:

$$\begin{aligned} f(x) &= x^3 - 3x + 1 \\ f'(x) &= 3x^2 - 3 \\ f'(1) &= 3(1)^2 - 3 \\ &= 3 - 3 \\ &= \boxed{0} \end{aligned}$$

7. Find the indicated derivative:

(a)
$$\begin{aligned} \frac{d}{dt}[16t^2] &= (2)(16t) \\ &= \boxed{32t} \end{aligned}$$

(b) $\frac{dC}{dr}$, *where* $C = 2\pi r$.

$$\begin{aligned} \frac{dC}{dr} &= \frac{d}{dr}(2\pi r) \\ &= 2\pi\frac{d}{dr}(r) \\ &= \boxed{2\pi} \end{aligned}$$

8. A spherical ballon is being inflated:

(a) Find the formula for the instantaneous rate of change of the volume V with respect to the radius r, given that $V = \frac{4}{3}\pi r^3$.
Solution: We are asked to find the change in V with respect to r, in other words, find $\frac{dV}{dr}$.

$$\frac{dV}{dr} = (3)\frac{4}{3}\pi r^2 = \boxed{4\pi r^2}$$

(b) Find the rate of change of V with respect to r at the instant when the radius is $r = 5$.
Solution:

$$\left.\frac{dV}{dr}\right|_{r=5} = 4\pi(5)^2 = 4\pi 25 = \boxed{100\pi}$$

9. Solve the following:

(a) *Find* $y'''(0)$, *where* $y = 4x^4 + 2x^3 + 3$

$$\begin{aligned} y' &= 16x^3 + 6x^2 \\ y'' &= 48x^2 + 12x \\ y''' &= 96x + 12 \\ y'''(0) &= 96(0) + 12 \\ &= \boxed{12} \end{aligned}$$

(b) *Find* $\left.\frac{d^4y}{dx^4}\right|_{x=1}$, *where* $y = \frac{6}{x^4}$

$$\begin{aligned} y &= 6x^{-4} \\ \frac{dy}{dx} &= (-4)6x^{-5} = -24x^{-5} \\ \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} &= (-5)(-24)x^{-6} \ = 120x^{-6} \\ \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} &= (-6)120x^{-7} = -720x^{-7} \\ \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d^4y}{dx^4} &= (-7)(-720)x^{-8} = 5040x^{-8} \\ \left.\frac{d^4y}{dx^4}\right|_{x=1} &= 5040(1)^{-8} \\ &= \boxed{5040} \end{aligned}$$

(c) *Show that* $y = x^3 + 3x + 1$ *satisfies* $y''' + xy'' - 2y' = 0$

i. First find the derivatives needed:

$$\begin{aligned} y &= x^3 + 3x + 1 \\ y' &= 3x^2 + 3 \\ y'' &= 6x \\ y''' &= 6 \end{aligned}$$

ii. Substitute:

$$\begin{aligned} y''' + xy'' - 2y' &= 6 + x(6x) - 2(3x^2 + 3) \\ &= 6 + 6x^2 - 6x^2 - 6 \\ &= \boxed{0} \end{aligned}$$

10. Find a function $y = ax^2 + bx + c$ whose graph has an x-intercept at 1, a y-intercept of -2, and a tangent line with a of slope of -1 at the y-intercept.

The y-intercept at -2 means that when $x = 0$, $y = -2$, i.e. $(0, -2)$ is on the graph. (*This is the easiest way to start because plugging in x=0 reduces the equation to find c immediately*)

$$\begin{aligned} y &= ax^2 + bx + c \\ -2 &= a(0)^2 + b(0) + c \\ -2 &= c \end{aligned}$$

So $y = ax^2 + bx - 2$. Now we use the x-intercept, which tells us that $(1, 0)$ is also on the graph, therefore,

$$\begin{aligned} 0 &= a(1)^2 + b(1) - 2 \\ 0 &= a + b - 2 \\ 2 &= a + b \end{aligned}$$

Now we also want a tangent line with a slope of -1 at the y-intercept. This means that $\frac{dy}{dx}\Big|_{x=0} = -1$.

$$\begin{aligned} y &= ax^2 + bx - 2 \\ \frac{dy}{dx} &= 2ax + b \\ \frac{dy}{dx}\Big|_{x=0} = -1 &= 2a(0) + b \\ -1 &= b \end{aligned}$$

Substitute $b = -1$ into $2 = a + b$ to get $a = 3$. Our function is thus, $\boxed{y = 3x^2 - x - 2}$

11. Find k if the curve (i) $y = x^2 + k$ is tangent to the line (ii) $y = 2x$.

Let P be the point at (x_0, y_0) at which (i) and (ii) are tangent to each other. Find dy/dx of (i):

$$y = x^2 + k$$
$$\frac{dy}{dx} = 2x \quad (iii)$$

We need the to find x_0 so that the slopes of (iii) and (ii) are equal at (x_0, y_0). The slope of the (ii) is clearly 2:

$$2x_0 = 2$$
$$x_0 = 1$$

Now P is on the line $y = 2x$, we can sub $x_0 = 1$ and obtain $y_0 = 2$. So $P = (1, 2)$. Plug P into (i):

$$y = x^2 + k$$
$$2 = 1^2 + k$$

$$\boxed{1 = k}$$

12. Find the x-coordinate of the point on the graph $y = x^2$ where the tangent line is parallel to the secant line that cuts the curve at $x = -1$ and $x = 2$.
(*Try drawing a picture to help visualize this problem.*)

The secant line is the line that goes through the points $(-1, 1)$ and $(2, 4)$, $(why?)$. The slope is then found by using the slope formula of two points:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{2 - (-1)} = \frac{3}{3} = 1.$$

We want this to be the same slope as the tangent line of $y = x^2$ which is $dy/dx = 2x$. Thus $2x = 1 \ \rightarrow \boxed{x = \frac{1}{2}}$ is the $x-$coordinate we are looking for.

13. Show that any two tangent lines to the parabola $y = ax^2,\ a \neq 0$, intersect at a point that is on the vertical line halfway between the points of tangency.

$$y = ax^2$$
$$\frac{dy}{dx} = 2ax$$

Let $P_1 = (x_1, y_1 = ax_1^2)$ and $P_2 = (x_2, y_2 = ax_2^2)$ be two different points of tangency. The tangent line at P_1 has the equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - ax_1^2 &= \left.\frac{dy}{dx}\right|_{x=x_1} (x - x_1) \\ y - ax_1^2 &= 2ax_1(x - x_1) \\ y - ax_1^2 &= (2ax)x_1 - 2ax_1^2 \\ y &= (2ax)x_1 - ax_1^2 \end{aligned}$$

Similarly, the tangent line at P_2 is $y = (2ax)x_2 - ax_2^2$.

Set $y = y$ and solve for x:

$$\begin{aligned} (2ax)x_1 - ax_1^2 &= (2ax)x_2 - ax_2^2 \\ (2ax)(x_1 - x_2) &= a(x_1^2 - x_2^2) \\ 2x(x_1 - x_2) &= (x_1 - x_2)(x_1 + x_2) \\ 2x &= (x_1 + x_2) \\ x &= \frac{1}{2}(x_1 + x_2) \end{aligned}$$

This is the x-coordinate of a point on the vertical line halfway between P_1 and $P - 2$.

14. Show that the segment of the tangent line to the graph of $y = \frac{1}{x},\ x > 0$, and the coordinate axes has an area of 2 square units.

Let $P_0 = (x_0, y_0)$ be a point on the curve. Then $y_0 = \frac{1}{x_0}$ and $\frac{dy}{dx} = -\frac{1}{x^2}$.

The equation for the tangent line is:

$$
\begin{aligned}
y - y_0 &= m(x - x_0)\\
y - \frac{1}{x_0} &= dy/dx|_{x=x_0}(x - x_0)\\
y - \frac{1}{x_0} &= -\frac{1}{x_0^2}(x - x_0)\\
y - \frac{1}{x_0} &= -\frac{x}{x_0^2} + \frac{1}{x_0}\\
y &= -\frac{x}{x_0^2} + \frac{2}{x_0}
\end{aligned}
$$

This line will intersect the x-axis when $y = 0$.

$$
\begin{aligned}
0 &= -\frac{x}{x_0^2} + \frac{2}{x_0}\\
\frac{x}{x_0^2} &= \frac{2}{x_0}\\
x &= 2x_0
\end{aligned}
$$

The line will intersect the y-axis when $x = 0$.

$$y = -\frac{0}{x_0^2} + \frac{2}{x_0} = \frac{2}{x_0}$$

So we have a triangle with coordinates: $\left(0, \frac{2}{x_0}\right)$, $(0, 0)$, $(2x_0, 0)$. The area of this triangle is: $\frac{1}{2}\left(\frac{2}{x_0}\right)(2x_0) = 2$.

15. Find conditions on a, b, c, d, so that the graph of the polynomial $f(x) = ax^3 + bx^2 + cx + d$ has exactly one horizontal tangent.

$f(x)$ will have a horizontal tangent when $f'(x) = 3ax^2 + 2bx + c = 0$. We can solve this by using the quadratic formula:

$$\begin{aligned} x &= \frac{-2b \pm \sqrt{(2b)^2 - 4(3a)c}}{2(3a)} \\ &= \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} \\ &= \frac{-2b \pm \sqrt{4(b^2 - 3ac)}}{6a} \\ &= \frac{-2b \pm 2\sqrt{b^2 - 3ac}}{6a} \\ &= \frac{-b \pm \sqrt{b^2 - 3ac}}{3a} \end{aligned}$$

Now x will have one real solution if $\sqrt{b^2 - 3ac} = 0$:

$$\begin{aligned} \left(\sqrt{b^2 - 3ac}\right)^2 &= 0^2 \\ b^2 - 3ac &= 0 \\ b^2 &= 3ac \\ \boxed{b = \pm\sqrt{3ac}} \end{aligned}$$

16. Let $f(x) = \begin{cases} x^2 - 16x, & x < 9 \\ \sqrt{x}, & x \geq 9 \end{cases}$

Is f continuous at $x = 9$? Determine whether f is differentiable at $x = 9$. If so, find the value of the derivative there.

$$\lim_{x \to 9^-} f(x) = (9)^2 - 16(9) = 81 - 144 = -63$$
$$\lim_{x \to 9^+} f(x) = \sqrt{9} = 3$$

Since $f(x)$ is not continuous at $x = 9$, it is not differentiable at $x = 9$. (*Differentiable implies continuous, therefore not continuous implies not differentiable.*)

17. Let $f(x) = \begin{cases} x^3 + \frac{1}{16}, & x < 1/2 \\ \frac{3}{4}x^2, & x \geq 1/2 \end{cases}$

Determine whether f is differentiable at $x = \frac{1}{2}$. If so, find the value of the derivative there.

$$\lim_{x \to \frac{1}{2}^-} f(x) = \left(\frac{1}{2}\right)^3 + \frac{1}{16} = \frac{1}{8} + \frac{1}{16} = \frac{3}{16}$$

$$\lim_{x \to \frac{1}{2}^+} f(x) = \frac{3}{4}\left(\frac{1}{2}\right)^2 = \frac{3}{4}\left(\frac{1}{4}\right) = \frac{3}{16}$$

Since these are equal, f has been shown to be continuous at $x = \frac{1}{2}$. (*If f was not continuous then we would immediately be able to say that f is not differentiable*) Now $f'(x)$ is given by:

$$f'(x) = \begin{cases} 3x^2, & x < 1/2 \\ \frac{3}{2}x. & x > 1/2 \end{cases}$$

Now we must still show differentiability at $x = \frac{1}{2}$:

$$\lim_{x \to \frac{1}{2}^-} f'(x) = 3\left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

$$\lim_{x \to \frac{1}{2}^+} f'(x) = \frac{3}{2}\left(\frac{1}{2}\right) = \frac{3}{4}$$

Therefore f is both continuous and differentiable at $x = \frac{1}{2}$ with $f'\left(\frac{1}{2}\right) = \frac{3}{4}$.