The Product and Quotient Rules
Selected Problems

Matthew Staley

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The Product and Quotient Rules: Selected Problems

1. Compute the derivative of the given function $f(x)$ by (i) multiplying and then differentiating and (ii) using the product rule. Verify that (i) and (ii) yield the same result.

   (a) $f(x) = (x^2 + 1)(x^2 - 1)$

      i.
$$\begin{aligned} f(x) &= x^4 - x^2 + x^2 - 1 \\ &= x^4 - 1 \\ f'(x) &= \boxed{4x^3} \end{aligned}$$

      ii.
$$\begin{aligned} f'(x) &= (x^2 + 1)\frac{d}{dx}(x^2 - 1) + (x^2 - 1)\frac{d}{dx}(x^2 + 1) \\ &= (x^2 + 1)(2x) + (x^2 - 1)(2x) \\ &= 2x^3 + 2x + 2x^3 - 2x \\ &= \boxed{4x^3} \end{aligned}$$

   (b) $f(x) = (x + 1)(x^2 - x + 1)$

      i.
$$\begin{aligned} f(x) &= x^3 - x^2 + x + x^2 - x + 1 \\ &= x^3 + 1 \\ f'(x) &= \boxed{3x^2} \end{aligned}$$

      ii.
$$\begin{aligned} f'(x) &= (x + 1)\frac{d}{dx}(x^2 - x + 1) + (x^2 - x + 1)\frac{d}{dx}(x + 1) \\ &= (x + 1)(2x - 1) + (x^2 - x + 1)(1) \\ &= 2x^2 - x + 2x - 1 + x^2 - x + 1 \\ &= \boxed{3x^2} \end{aligned}$$

2. Find $f'(x)$ for the given $f(x)$:

(a)

$$f(x) = (x^3 + 7x^2 - 8)(2x^{-3} + x^{-4})$$

$$f'(x) = (x^3 + 7x^2 - 8)\frac{d}{dx}(2x^{-3} + x^{-4}) + (2x^{-3} + x^{-4})\frac{d}{dx}(x^3 + 7x^2 - 8)$$

$$= (x^3 + 7x^2 - 8)(-6x^{-4} - 4x^{-5}) + (2x^{-3} + x^{-4})(3x^2 + 14x)$$

$$= x^3(-6x^{-4} - 4x^{-5}) + 7x^2(-6x^{-4} - 4x^{-5}) - 8(-6x^{-4} - 4x^{-5}) + 2x^{-3}(3x^2) + 2x^{-3}(14x) + x^{-4}(3x^2) + x^{-4}(14x)$$

$$= -6x^{-1} - 4x^{-2} - 42x^{-2} - 28x^{-3} + 48x^{-4} + 32x^{-5} + 6x^{-1} + 28x^{-2} + 3x^{-2} + 14x^{-3}$$

$$= x^{-1}(-6 + 6) + x^{-2}(3 - 4 + 28 - 42) + x^{-3}(14 - 28) + 48x^{-4} + 32x^{-5}$$

$$= \boxed{-15x^{-2} - 14x^{-3} + 48x^{-4} + 32x^{-5}}$$

(b)

$$f(x) = \left(\frac{1}{x} + \frac{1}{x^2}\right)(3x^3 + 27) \quad (\textit{First multiply then differentiate})$$

$$= 3x^2 + 27x^{-1} + 3x + 27x^{-2}$$

$$f'(x) = 6x - 27x^{-2} + 3 - 54x^{-3}$$

$$= \boxed{3 + 6x - 27x^{-2} - 54x^{-3}}$$

3. Find $dy/dx|_{x=1}$:

(a)

$$\begin{aligned} y &= \frac{3}{\sqrt{x}+2} \\ \frac{dy}{dx} &= \frac{(\sqrt{x}+2)\frac{d}{dx}(3) \; - \; 3\frac{d}{dx}(\sqrt{x}+2)}{(\sqrt{x}+2)^2} \\ &= \frac{(\sqrt{x}+2)(0) - \; 3(\frac{1}{2}x^{-1/2})}{(\sqrt{x}+2)^2} \\ &= \frac{\frac{-3}{2\sqrt{x}}}{(\sqrt{x}+2)^2} \end{aligned}$$

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{x=1} &= \frac{\frac{-3}{2\sqrt{1}}}{(\sqrt{1}+2)^2} \\ &= \frac{\frac{-3}{2}}{3^2} \\ &= \left(\frac{-3}{2}\right)\left(\frac{1}{9}\right) = \boxed{-\frac{1}{6}} \end{aligned}$$

(b)

$$\begin{aligned} y &= \frac{4x+1}{x^2-5} \\ \frac{dy}{dx} &= \frac{(x^2-5)\frac{d}{dx}(4x+1) \; - \; (4x+1)\frac{d}{dx}(x^2-5)}{(x^2-5)^2} \\ &= \frac{(x^2-5)(4) - (4x+1)(2x)}{(x^2-5)^2} \\ &= \frac{4x^2-20-8x^2-2x}{(x^2-5)^2} \\ &= \frac{-4x^2-2x-20}{(x^2-5)^2} \end{aligned}$$

$$
\begin{aligned}
\left.\frac{dy}{dx}\right|_{x=1} &= \frac{-4(1)^2 - 2(1) - 20}{((1)^2 - 5)^2} \\
&= \frac{-26}{(-4)^2} \\
&= -\frac{26}{16} = \boxed{-\tfrac{13}{8}}
\end{aligned}
$$

4. Find $g'(4)$ given that $f(4) = 3$ and $f'(4) = -5$:

(a)

$$
\begin{aligned}
g(x) &= \sqrt{x} f(x) \\
g'(x) &= f'(x)\sqrt{x} + \frac{1}{2\sqrt{x}} f(x) \\
g'(4) &= f'(4)\sqrt{4} + \frac{1}{2\sqrt{x}} f(4) \\
&= (-5)(2) + \frac{1}{2(2)}(3) \\
&= -10 + \frac{3}{4} \\
&= \frac{-40}{4} + \frac{3}{4} = \boxed{\tfrac{-37}{4}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
g(x) &= \frac{f(x)}{x} \\
g'(x) &= \frac{x f'(x) - (1) f(x)}{x^2} \\
g'(4) &= \frac{4 f'(4) - f(4)}{4^2} \\
&= \frac{4(-5) - 3}{16} = \boxed{\tfrac{-23}{16}}
\end{aligned}
$$

5. Find $F'(\pi)$ given that $f(\pi) = 10,\ f'(\pi) = -1,\ g(\pi) = -3,\ and\ g'(\pi) = 2.$

(a)

$$\begin{aligned} F(x) &= 6f(x) - 5g(x) \\ F'(x) &= 6f'(x) - 5g'(x) \\ F'(\pi) &= 6f'(\pi) - 5g'(\pi) \\ &= 6(-1) - 5(2) \\ &= -6 - 10 = \boxed{\text{-16}} \end{aligned}$$

(b)

$$\begin{aligned} F(x) &= x(f(x) + g(x)) \\ F'(x) &= f(x) + g(x) + x(f'(x) + g'(x)) \\ F'(\pi) &= f(\pi) + g(\pi) + x(f'(\pi) + g'(\pi)) \\ &= 10 - 3 + \pi(-1 + 2) = \boxed{7 + \pi} \end{aligned}$$

(c)

$$\begin{aligned} F(X) &= 2f(x)g(x) \\ F'(x) &= 2f(x)g'(x) + 2g(x)f'(x) \\ F'(\pi) &= 2f(\pi)g'(\pi) + 2g(\pi)f'(\pi) \\ &= 2(10)(2) + 2(-3)(-1) \\ &= 40 + 6 = \boxed{46} \end{aligned}$$

(d)

$$\begin{aligned} F(x) &= \frac{f(x)}{4 + g(x)} \\ F'(x) &= \frac{(4 + g(x))f'(x) - f(x)g'(x)}{(4 + g(x))^2} \\ F'(\pi) &= \frac{(4 + g(\pi))f'(\pi) - f(\pi)g'(\pi)}{(4 + g(\pi))^2} \\ &= \frac{(4 - 3)(-1) - (10)(2)}{(4 - 3)^2} \\ &= \frac{-1 - 20}{1^2} = \boxed{21} \end{aligned}$$

6. Find all values of x at which the tangent line to the given curve satisfies the stated property:

(a)
$$\begin{aligned} y &= \frac{x^2-1}{x+2}; \quad \textit{Horizontal} \\ \frac{dy}{dx} &= \frac{(2x)(x+2)-(1)(x^2-1)}{(x+2)^2} \\ &= \frac{2x^2+4x-x^2+1}{(x+2)^2} \\ &= \frac{x^2+4x+1}{(x+2)^2} \end{aligned}$$

Now the slope of the tangent line is horizontal whenever $dy/dx = 0$. This is occurs only when the numerator $x^2+4x+1=0$. We cannot factor this equation, so by the quadratic formula,

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2-4(1)(1)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{16-4}}{2} \\ &= \frac{-4 \pm \sqrt{12}}{2} \\ &= \frac{-4 \pm 4\sqrt{3}}{2} = \boxed{-2 \pm 2\sqrt{3}} \end{aligned}$$

So $x = -2+2\sqrt{3}$ and $x = -2-2\sqrt{3}$ are the values at which the tangent line is horizontal.

(b)
$$\begin{aligned} y &= \frac{x^2+1}{x+1}; \quad \textit{Parallel to the line } y = x \\ \frac{dy}{dx} &= \frac{2x(x+1)-(x^2+1)}{(x+1)^2} \\ &= \frac{2x^2+2x-x^2-1}{(x+1)^2} \\ &= \frac{x^2+2x-1}{(x+1)^2} \end{aligned}$$

The tangent line is parallel to the line $y = x$ when it has slope 1. So set $dy/dx = 1$ and solve for x :

$$\begin{aligned}
\frac{x^2+2x-1}{(x+1)^2} &= 1 \\
x^2+2x-1 &= (x+1)^2 \\
x^2+2x-1 &= x^2+2x+1 \\
-1 &= 1
\end{aligned}$$

This is a contradiction and so this implies that the tangent line is never parallel to the line $y = x$ for any x values!

(c) $$y = \frac{x+3}{x+2}; \quad \textit{Perpendicular to the line } y = x.$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1(x+2)-1(x+3)}{(x+2)^2} \\
&= \frac{x+2-x-3}{(x+2)^2} \\
&= \frac{-1}{(x+2)^2}
\end{aligned}$$

The tangent line is perpendicular to the line $y = x$ when it has slope -1. Set $dy/dx = -1$ and solve for x :

$$\begin{aligned}
\frac{-1}{(x+2)^2} &= -1 \\
1 &= (x+2)^2 \\
1 &= (x^2+4x+4) \\
0 &= x^2+4x+3 \\
0 &= (x+3)(x+1) \\
&\boxed{x = -3, -1}
\end{aligned}$$

$x = -3$ and $x = -1$ are the values at which the tangent line is perpendicular to the line $y = x$.

To find the actual points we plug in the found $x - values$ into y.

When $x = -3$, $y = \frac{-3+3}{-3+2} = 0$
When $x = -1$, $y = \frac{-1+3}{-1+2} = 2$

The tangent line is perpendicular to the line $y = x$ at the points $(-3, 0),\ (-1, 2)$.

(d)
$$\begin{aligned} y &= \frac{1}{x+4}; \quad \textit{Passes through the origin.} \\ \frac{dy}{dx} &= \frac{(x+4)(0) - 1(1)}{(x+4)^2} \\ &= \frac{-1}{(x+4)^2} \end{aligned}$$

We want to find a fixed value of x, say x_0, so that the tangent line of y at the point $\left(x_0, \frac{1}{x_0+4}\right)$ goes through $(0, 0)$. So we have two points and a slope $dy/dx|_{x=x_0}$. Using the point slope formula we can substitute in our values and solve for x_0:

$$\begin{aligned} y_1 - y_2 &= m(x_1 - x_2) \\ \frac{1}{x_0+4} - 0 &= \frac{-1}{(x_0+4)^2}(x_0 - 0) \\ \frac{1}{x_0+4} &= \frac{-x_0}{(x_0+4)^2} \\ x_0 + 4 &= -x_0 \\ 4 &= -2x_0 \\ \boxed{-2 = x_0} \end{aligned}$$

So $\left(-2, \frac{1}{2}\right)$ is the only point at which the tangent line passes through the origin.

(e)
$$y = \frac{2x+5}{x+2}; \quad y - intercept\ 2.$$
$$\begin{aligned}\frac{dy}{dx} &= \frac{(x+2)(2)-(2x+5)(1)}{(x+2)^2} \\ &= \frac{2x+4-2x-5}{(x+2)^2} \\ &= \frac{-1}{(x+2)^2}\end{aligned}$$

Fix x_0. We have the points $\left(x_0, y_0 = \frac{2x_0+5}{x_0+2}\right)$, the $y - intercept$ $(0,2)$, and the slope $dy/dx|_{x=x_0}$. Substitute these into the point-slope formula and solve for x_0:

$$y_1 - y_2 = m(x_1 - x_2)$$

$$\frac{2x_0+5}{x_0+2} - 2 = -\frac{1}{(x_0+2)^2}(x_0 - 0)$$

$$\frac{2x_0+5}{x_0+2} - \frac{2(x_0+2)}{x_0+2} = \frac{-x_0}{(x_0+2)^2}$$

$$\frac{2x_0+5-2x_0-4}{x_0+2} = \frac{-x_0}{(x_0+2)^2}$$

$$\frac{1}{x_0+2} = \frac{-x_0}{(x_0+2)^2}$$

$$\begin{aligned}x_0 + 2 &= -x_0 \\ 2 &= -2x_0\end{aligned}$$

$$\boxed{-1 = x_0}$$

So $(-1,3)$ is the only point at which the tangent line passes through the origin.

7. Apply the product rule twice to show that if f, g, and h are differentiable functions, then $f \cdot g \cdot h$ is differentiable, and

$$(f \cdot g \cdot h)' = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$$

(I will suppress the use of $\cdot$ so that $f \cdot g \cdot h = fgh$)
First, since f and g are differentiable, then the product $(f)(g) = fg$ is differentiable (This is the product rule theorem). Since fg and h are differentiable, then the product $(fg)h = fgh$ is differentiable again by the product rule theorem. So fgh is a differentiable and:

$$\begin{aligned}(fgh)' &= ((fg)h)' \\ &= (fg)h' + (fg)'h \\ &= fgh' + h(f'g + g'f) \\ &= fgh' + hf'g + hg'f \\ &= f'gh + fg'h + fgh'\end{aligned}$$

8. Use the above formula to find the following derivatives:

(a) $\frac{d}{dx}\left[(2x+1)(1+\frac{1}{x})(x^{-3}+7)\right]$

$= \left(\frac{d}{dx}(2x+1)\right)(1+x^{-1})(x^{-3}+7) + (2x+1)\left(\frac{d}{dx}(1+x^{-1})\right)(x^{-3}+7) +$

$\ldots\ (2x+1)(1+x^{-1})\left(\frac{d}{dx}(x^{-3}+7)\right)$

$= \boxed{2(1+x^{-1})(x^{-}2) + (2x+1)(-x^{-2})(x^{-3}+7) + (2x+1)(1+x^{-1})(-3x^{-4})}$

(b) $\frac{d}{dx}\left[(x^7+2x-3)^3\right]$

$$= \frac{d}{dx}\left[(x^7+2x-3)(x^7+2x-3)(x^7+2x-3)\right]$$

$$= \left(\frac{d}{dx}(x^7+2x-3)\right)(x^7+2x-3)(x^7+2x-3)$$

$$\cdots + (x^7+2x-3)\left(\frac{d}{dx}(x^7+2x-3)\right)(x^7+2x-3)$$

$$\cdots + (x^7+2x-3)(x^7+2x-3)\left(\frac{d}{dx}(x^7+2x-3)\right)$$

$$= (7x^6+2)(x^7+2x-3)(x^7+2x-3)$$

$$\cdots + (x^7+2x-3)(7x^6+2)(x^7+2x-3)$$

$$\cdots + (x^7+2x-3)(x^7+2x-3)(7x^6+2)$$

$$= (7x^6+2)(x^7+2x-3)^2 + (7x^6+2)(x^7+2x-3)^2 + (7x^6+2)(x^7+2x-3)^2$$

$$= \boxed{3(7x^6+2)(x^7+2x-3)^2}$$