Derivatives of Trig Functions
Selected Problems

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Derivatives of Trig Functions: Selected Problems

1. Find $f'(x)$:

(a)

$$f(x) = 4\cos x + 2\sin x$$

$$\begin{aligned} f'(x) &= 4\frac{d}{dx}(\cos x) + 2\frac{d}{dx}(\sin x) \\ &= \boxed{-4\sin x + 2\cos x} \end{aligned}$$

(b)

$$f(x) = 2\sin^2 x = 2\left(\sin x\right)^2 = 2\sin x \sin x$$

$$\begin{aligned} f'(x) &= 2\frac{d}{dx}(\sin x \sin x) \\ &= 2(\sin x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(\sin x)) \\ &= 2(\sin x\cos x + \sin x\cos x) \\ &= 2(2\sin x\cos x) \\ &= \boxed{4\sin x\cos x} \end{aligned}$$

(c)

$$f(x) = \sec x - \sqrt{2}\tan x$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\sec x) - \sqrt{2}\frac{d}{dx}(\tan x) \\ &= \boxed{\sec x\tan x - \sqrt{2}\sec^2 x} \end{aligned}$$

(d)

$$f(x) = \frac{5 - \cos x}{5 + \sin x}$$

$$f'(x) = \frac{(5 + \sin x)\frac{d}{dx}(5 - \cos x) - (5 - \cos x)\frac{d}{dx}(5 + \sin x)}{(5 + \sin x)^2}$$

$$= \frac{(5 + \sin x)(\sin x) - (5 - \cos x)(\cos x)}{(5 + \sin x)^2}$$

$$= \frac{5\sin x + \sin^2 x - 5\cos x + \cos^2 x}{(5 + \sin x)^2}$$

$$= \frac{(\sin^2 x + \cos^2 x) + 5(\sin x - \cos x)}{(5 + \sin x)^2}$$

$$= \boxed{\tfrac{1+5(\sin x - \cos x)}{(5+\sin x)^2}}$$

(e)

$$f(x) = \sec x \tan x$$

$$f'(x) = \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x)$$

$$= \sec x(\sec^2 x) + \tan x(\sec x \tan x)$$

$$= \boxed{\sec^3 x + \sec x \tan^2 x}$$

2. Find $\frac{d^2y}{dx^2}$:

(a)

$$
\begin{aligned}
y &= x\cos x \\[1em]
\frac{dy}{dx} &= \cos x \frac{d}{dx}(x) + x\frac{d}{dx}(\cos x) \\
&= \cos x - x\sin x \\[1em]
\frac{d^2y}{dx^2} &= \frac{d}{dx}(\cos x) - \frac{d}{dx}(x\sin x) \\
&= -\sin x - (\sin x \frac{d}{dx}(x) + x\frac{d}{dx}(\sin x)) \\
&= -\sin x - (\sin x + x\cos x) \\
&= -\sin x - \sin x - x\cos x \\
&= \boxed{-2\sin x - x\cos x}
\end{aligned}
$$

(b)

$$
\begin{aligned}
y &= \csc x \\[1em]
\frac{dy}{dx} &= \frac{d}{dx}(\csc x) \\
&= -\csc x\cot x \\[1em]
\frac{d^2y}{dx^2} &= -\frac{d}{dx}(\csc x\cot x) \\
&= -(\cot x \frac{d}{dx}(\csc x) + \csc x\frac{d}{dx}(\cot x)) \\
&= -(\cot x(-\csc x\cot x) + \csc x(-\csc^2 x)) \\
&= \boxed{\cot^2 x\csc x + \csc^3 x}
\end{aligned}
$$

3. Find the equation of the line tangent to the graph of $\sin x$ at $x = \pi/4$:

Let $f(x) = \sin x$, then $f(\frac{\pi}{4}) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.
Now $f'(x) = \cos x$, so $f'(\frac{\pi}{4}) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.
We now have a point, $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$, and a slope, $\frac{1}{\sqrt{2}}$. So plugging this into the point-slope formula we obtain our line:

$$y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$
$$y - \frac{1}{\sqrt{2}} = \frac{x}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}$$

$$\boxed{y = \frac{x}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}}$$

4. Show that $y = \cos x$ and $y = \sin x$ are solutions to the equation $y'' + y = 0$.

$$y = \cos x$$
$$y' = -\sin x$$
$$y'' = -\cos x$$

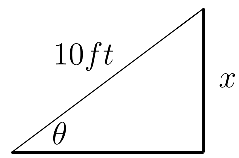
$$y'' + y = -\cos x + \cos x = 0.$$

$$y = \sin x$$
$$y' = \cos x$$
$$y'' = -\sin x$$

$$y'' + y = -\sin x + \sin x = 0.$$

5. A 10-ft ladder leans against a wall at an angle θ. The of top of the ladder is x feet above the ground. If the bottom of the ladder is pushed towards the wall, find the rate at which x changes with respect to θ when $\theta = 60°$. Express the answer in units of feet/degree.

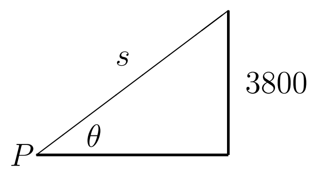
$$\sin\theta = \frac{x}{10}$$
$$x = 10\sin\theta$$



$$\frac{dx}{d\theta} = 10\frac{d}{d\theta}(\sin\theta) = 10\cos\theta$$

$$\left.\frac{dx}{d\theta}\right|_{\theta=60°=\frac{\pi}{3}} = 10\cos\left(\frac{\pi}{3}\right) = 10(\frac{1}{2}) = 5$$
$$= \frac{5ft}{rad}\left(\frac{\pi\ rad}{180°}\right)$$
$$= \boxed{\frac{\pi ft}{36°} \approx 0.087\frac{ft}{deg}}$$

6. An airplane is flying on a horizontal path at a height of 3800 ft. At what rate is the distance s between the airplane and the point P changing with respect to θ when $\theta = 30°$? Express the answer in units of feet/degree.



$$\sin\theta = \frac{3800}{s}$$

$$s = \frac{3800}{\sin\theta} = 3800\csc\theta$$

$$\frac{ds}{d\theta} = 3800\frac{d}{d\theta}(\csc\theta) = -3800\csc\theta\cot\theta$$

$$\left.\frac{ds}{d\theta}\right|_{\theta=30°=\frac{\pi}{6}} = -3800\csc\left(\frac{\pi}{6}\right)\cot\left(\frac{\pi}{6}\right)$$

$$= -3800(2)(\sqrt{3}) = -7600\sqrt{3}$$

$$= -7600\sqrt{3}\,\frac{rad}{ft}\left(\frac{\pi\ rad}{180°}\right)$$

$$= \boxed{\frac{-380\sqrt{3}\pi}{9}\frac{ft}{deg} \approx -230\frac{ft}{deg}}$$

7. Determine where f is differentiable.

(a) $f(x) = \sin x$

$\sin x$ is differentiable for all real values of x because it is continuous for all real values of x.

(b) $f(x) = \cot x$

$\cot x = \frac{\cos x}{\sin x}$ is not continuous whenever $\sin x = 0$, or $x = 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots$

So $f(x)$ is differentiable for all $x \neq\ n\pi,\ n = 0, \pm 1,\ \pm 2,\ \pm 3,\ \ldots$

(c) $f(x) = \frac{1}{1+\cos x}$

This is not continuous whenever

$$\begin{aligned} 1 + \cos x &= 0 \\ \cos x &= -1 \\ x &= \pi, 3\pi, 5\pi, \ldots \end{aligned}$$

So $f(x)$ is differentiable for all $x \neq (2n+1)\pi,\ n = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \ldots$