The Chain Rule
Selected Problems

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The Chain Rule: Selected Problems

1. Given that $f'(9) = 5,\ g(2) = 9$, and $g'(2) = -3$, find $(f \circ g)'(2)$.

$$\begin{aligned}(f \circ g)'(2) &= f'(g(2))g'(2)\\ &= f'(9)(-3)\\ &= 5(-3) = \boxed{\text{-15}}\end{aligned}$$

2. Let $f(x) = 5\sqrt{x}$ and $g(x) = 4 + \cos x$.

(a) Find $(f \circ g)(x)$ and $(f \circ g)'(x)$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(4 + \cos x)\\ &= \boxed{5\sqrt{4 + \cos x}}\end{aligned}$$

$$\begin{aligned}(f \circ g)'(x) &= f'(g(x))g'(x), \ \left(\mathit{Note}\ f'(x) = \frac{5}{2\sqrt{x}}\right)\\ &= \frac{5}{2\sqrt{g(x)}}(-\sin x)\\ &= \boxed{-\frac{5\sin x}{2\sqrt{4 + \cos x}}}\end{aligned}$$

(b) Find $(g \circ f)(x)$ and $(g \circ f)'(x)$.

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(5\sqrt{x})\\ &= \boxed{4 + \cos\left(5\sqrt{x}\right)}\end{aligned}$$

$$\begin{aligned}(g \circ f)'(x) &= g'(f(x))f'(x)\\ &= -\sin\left(f(x)\right)\frac{5}{2\sqrt{x}}\\ &= \boxed{-\sin\left(5\sqrt{x}\right)\frac{5}{2\sqrt{x}}}\end{aligned}$$

3. Find $f'(x)$ for the following $f(x)$:

(a)

$$\begin{aligned}
f(x) &= (3x^2+2x-1)^6 \\
f'(x) &= 6(3x^2+2x-1)^5 \frac{d}{dx}(3x^2+2x-1) \\
&= 6(3x^2+2x-1)(6x+2) \\
&= \boxed{12(3x^2+2x-1)(3x+1)}
\end{aligned}$$

(b)

$$\begin{aligned}
f(x) &= \left(x^3 - \frac{7}{x}\right)^{-2} \\
f'(x) &= (-2)\left(x^3 - \frac{7}{x}\right)^{-3} \frac{d}{dx}\left(x^3 - \frac{7}{x}\right) \\
&= (-2)\left(x^3 - \frac{7}{x}\right)^{-3} \left(3x^2 - \left(-\frac{7}{x^2}\right)\right) \\
&= \boxed{(-2)\left(x^3 - \frac{7}{x}\right)^{-3} \left(3x^2 + \frac{7}{x^2}\right)}
\end{aligned}$$

(c)

$$\begin{aligned}
f(x) &= \sqrt{x^3-2x+5} \\
f'(x) &= \frac{1}{2}\left(x^3-2x+5\right)^{-1/2} \frac{d}{dx}\left(x^3-2x+5\right) \\
&= \boxed{\frac{3x^2-2}{2\sqrt{x^3-2x+5}}}
\end{aligned}$$

(d)

$$\begin{aligned}
f(x) &= \tan\left(\sqrt{x}\,\right) \\
f'(x) &= \sec^2\left(\sqrt{x}\,\right) \frac{d}{dx}\left(\sqrt{x}\right) \\
&= \sec^2\left(\sqrt{x}\,\right)\left(\frac{1}{2\sqrt{x}}\right) \\
&= \boxed{\frac{\sec^2\left(\sqrt{x}\,\right)}{2\sqrt{x}}}
\end{aligned}$$

4. Find dy/dx:

(a)

$$\begin{aligned}
y &= x^3 \sin^2 5x \\
\frac{dy}{dx} &= x^3 \frac{d}{dx}\left(\sin^2(5x)\right) + \sin^2(5x)\frac{d}{dx}(x^3) \\
&= x^3\left(2\sin(5x)\frac{d}{dx}(5x)\right) + \sin^2(5x)(3x^2) \\
&= \boxed{10x^3\sin(5x) + 3x^2\sin^2(5x)}
\end{aligned}$$

(b)

$$\begin{aligned}
y &= \sin(\tan(3x)) \\
\frac{dy}{dx} &= \cos(\tan(3x))\frac{d}{dx}(\tan(3x)) \\
&= \cos(\tan(3x))\sec^2(3x)\frac{d}{dx}(3x) \\
&= \boxed{3\sec^2(3x)\cos(\tan(3x))}
\end{aligned}$$

(c)

$$\begin{aligned}
y &= \frac{(2x+3)^3}{(4x^2-1)^8} \\
\frac{dy}{dx} &= \frac{(4x^2-1)^8\frac{d}{dx}(2x+3)^3 - (2x+3)^3\frac{d}{dx}(4x^2-1)^8}{((4x^2-1)^8)^2} \\
&= \frac{(4x^2-1)^8 3(2x+3)^2\frac{d}{dx}(2x) - (2x+3)^3 8(4x^2-1)^7\frac{d}{dx}(4x^2-1)}{(4x^2-1)^{16}} \\
&= \frac{(4x^2-1)^8 3(2x+3)^2(2) - (2x+3)^3 8(4x^2-1)^7(8x)}{(4x^2-1)^{16}} \\
&= \frac{2(4x^2-1)^7(2x+3)^2\left[3(4x^2-1) - 32x(2x+3)\right]}{(4x^2-1)^{16}} \\
&= \frac{2(2x+3)^2[12x^2-3-64x^2-96x]}{(4x^2-1)^{16-7}} \\
&= \frac{2(2x+3)^2[-52x^2-96x-3]}{(4x^2-1)^9} \\
&= \boxed{-\frac{2(2x+3)^2(52x^2+96x+3)}{(4x^2-1)^9}}
\end{aligned}$$

5. Find an equation for the tangent line to the graph at the specified value of x.

(a) $y = \sin(1 + x^3), \quad x = -3.$

First we need a point. So when $x = -3$,

$$\begin{aligned} y &= \sin(1 + (-3)^3) \\ &= \sin(1 - 27) \\ &= \sin(-26) \\ &= -\sin(26) \end{aligned}$$

Second, we need a slope at this point, i.e. $dy/dx|_{x=-3}$,

$$y = \sin\left(1 + x^3\right)$$

$$\begin{aligned} \frac{dy}{dx} &= \cos(1 + x^3)\frac{d}{dx}(1 + x^3) \\ &= \cos(1 + x^3)(3x^2) \end{aligned}$$

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{x=-3} &= \cos(1 + (-3)^3)(3(-3)^2) \\ &= \cos(1 - 27)(27) \\ &= 27\cos(-26) \\ &= 27\cos(26) \end{aligned}$$

Now plug into the point-slope formula:

$$\begin{aligned} y - (-\sin(26)) &= 27\cos(26)(x - (-3)) \\ y + \sin(26) &= 27x\cos(26) + 81\cos(26) \\ y &= 27x\cos(26) + 81\cos(26) - \sin(26) \end{aligned}$$

The equation of the tangent line is $\boxed{y = 27\cos(26)x + 81\cos(26) - \sin(26)}$.

(b) $y = \left(x - \frac{1}{x}\right)^3, \quad x = 2.$

When $x = 2$,

$$y = \left(2 - \frac{1}{2}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

Now find $dy/dx_{x=2}$,

$$y = \left(x - \frac{1}{x}\right)^3 = \left(x - x^{-1}\right)^3$$

$$\begin{aligned} \frac{dy}{dx} &= 3\left(x - x^{-1}\right)^2 \frac{d}{dx}\left(x - x^{-1}\right) \\ &= 3(x - x^{-1})^2(1 - (-1x^{-2})) \\ &= 3\left(x - \frac{1}{x}\right)^2\left(1 + \frac{1}{x^2}\right) \end{aligned}$$

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{x=2} &= 3\left(2 - \frac{1}{2}\right)^2\left(1 + \frac{1}{2^2}\right) \\ &= 3\left(\frac{3}{2}\right)^2\left(\frac{5}{4}\right) \\ &= 3\left(\frac{9}{4}\right)\left(\frac{5}{4}\right) = \frac{135}{16} \end{aligned}$$

Plug into the point-slope formula:

$$y - \frac{27}{8} = \frac{135}{16}(x - 2)$$

$$y - \frac{27}{8} = \frac{135}{16}x - \frac{135}{8}$$

$$y = \frac{135}{16}x - \frac{135}{8} + \frac{27}{8}$$

$$y = \frac{135}{16}x - \frac{108}{8}$$

$$y = \frac{135}{16}x - \frac{27}{2}$$

The equation of the tangent line is $\boxed{y = \frac{135}{16}x - \frac{27}{2}}$.