

Related Rates
Selected Problems

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1. Both x and y denote functions of t that are related by the given equation. Use this equation and the given derivative information to find the specified derivative.

- (a) $x + 4y = 3$; Given that $dx/dt = 1$, find dy/dt when $x = 2$.

$$\begin{aligned}x + 4y &= 3 \\ \frac{d}{dt}(x + 4y) &= \frac{d}{dt}(3) \\ \frac{dx}{dt} + 4\frac{dy}{dt} &= 0 \\ 1 + 4\frac{dy}{dt} &= 0\end{aligned}$$

$$\boxed{\frac{dy}{dt} = -\frac{1}{4}}$$

- (b) $y = 3x + 5$; Given that $dy/dt = -1$, find dx/dt when $x = 0$.

$$\begin{aligned}y &= 3x + 5 \\ \frac{dy}{dt} &= 3\frac{dx}{dt} \\ -1 &= 3\frac{dx}{dt}\end{aligned}$$

$$\boxed{\frac{dx}{dt} = -\frac{1}{3}}$$

(c) $4x^2 + 9y^2 = 1$; Given that $dx/dt = 3$, find dy/dt when $(x, y) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)$.

$$\begin{aligned}4x^2 + 9y^2 &= 1 \\ \frac{d}{dt}(4x^2 + 9y^2) &= \frac{d}{dt}(1) \\ 8x \frac{dx}{dt} + 18y \frac{dy}{dt} &= 0 \\ 8 \left(\frac{1}{2\sqrt{2}}\right) \cdot 3 + 18 \left(\frac{1}{3\sqrt{2}}\right) \frac{dy}{dt} &= 0 \\ \frac{6}{\sqrt{2}} \frac{dy}{dt} &= -\frac{12}{\sqrt{2}} \\ \frac{dy}{dt} &= -\frac{12}{\sqrt{2}} \cdot \frac{\sqrt{2}}{6} \\ \boxed{\frac{dy}{dt} = -2}\end{aligned}$$

2. Let A be the area of a square whose sides have length x , and assume that x varies with time t .

(a) Write an equation that relates A to x .

$$A = x^2$$

(b) Use this equation to find an equation that relates dA/dt and dx/dt

$$\begin{aligned}A &= x^2 \\ \frac{d}{dt}(A) &= \frac{d}{dt}(x^2) \\ \frac{dA}{dt} &= 2x \frac{dx}{dt}\end{aligned}$$

(c) At a certain instant the sides are 3 ft long and increasing at a rate of 2 ft/min. How fast is the area increasing at that instant?

(Find $\left.\frac{dA}{dt}\right|_{x=3}$ given that $\left.\frac{dx}{dt}\right|_{x=3} = 2$)

$$\left.\frac{dA}{dt}\right|_{x=3} = 2(3) \left.\frac{dx}{dt}\right|_{x=3} = 6(2) = \boxed{12 \text{ ft}^2/\text{min}}$$

3. Let V be the volume of a cylinder having height h and radius r , and assume that h and r vary with time.

(a) How are dV/dt , dh/dt , and dr/dt related?

$$V = \pi r^2 h$$

$$\frac{d}{dt}(V) = \frac{d}{dt}(\pi r^2 h)$$

$$\frac{dV}{dt} = \pi \left(r^2 \frac{dh}{dt} + 2rh \frac{dr}{dt} \right)$$

- (b) At a certain instant, the height is 6 in and increasing at 1 in/s, while the radius is 10 in and decreasing at 1 in/s. How fast is the volume changing at that instant? Is the volume increasing or decreasing at that instant?

$$\left. \frac{dV}{dt} \right|_{r=10}^{h=6} = \pi \left((10)^2 \left. \frac{dh}{dt} \right|_{r=10}^{h=6} + 2(10)(6) \left. \frac{dr}{dt} \right|_{r=10}^{h=6} \right)$$

$$= \pi(100(+1) + 120(-1))$$

$$= \boxed{-20\pi \text{ in}^3/\text{s}}$$

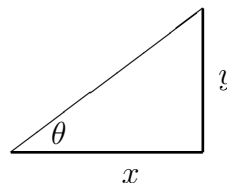
$\frac{dV}{dt}$ is negative, so the volume is decreasing.

4. Let θ (in radians) be an acute angle in a right triangle, and let x and y , respectively, be the lengths and sides adjacent to and opposite θ . Suppose also that x and y vary with time. How are $\frac{d\theta}{dt}$, dx/dt , and dy/dt related?

$$\tan \theta = \frac{y}{x}$$

$$\frac{d}{dt}(\tan(\theta)) = \frac{d}{dt} \left(\frac{y}{x} \right)$$

$$\sec^2(\theta) \frac{d\theta}{dt} = \frac{x \frac{dy}{dt} - y \frac{dx}{dt}}{x^2}$$



$$\boxed{\frac{d\theta}{dt} = \frac{\cos^2(\theta)}{x^2} \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right)}$$

5. The minute hand of a certain clock is 4 in long. Starting from the moment when the hand is pointing straight up, how fast is the area of the sector that is swept out by the hand increasing at any instant during the next revolution of the hand?

The area of a semi-circle is given by the formula: $A = \frac{1}{2}\theta r^2$. Our radius is 4 in so, $A = \frac{1}{2}\theta(4)^2 = 8\theta$. So the change in area at any instant is given by

$$\boxed{\frac{dA}{dt} = 8 \frac{d\theta}{dt}}$$

6. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of $6 \text{ mi}^2/h$. How fast is the radius of the spill increasing when the area is 9 mi^2 ?

We are given $\frac{dA}{dt} = 6$ and we are asked to find $\frac{dr}{dt}|_{A=9}$. However, we need to figure out what the radius is that gives us an area of 9 mi^2 .

$$\begin{aligned} A &= \pi r^2 \\ 9 &= \pi r^2 \\ \frac{9}{\pi} &= r^2 \\ \frac{3}{\sqrt{\pi}} &= r \end{aligned}$$

Now we need to find an equation that relates dA/dt and dr/dt .

$$\begin{aligned} A &= \pi r^2 \\ \frac{d}{dt}(A) &= \frac{d}{dt}(\pi r^2) \\ \frac{dA}{dt} &= \pi 2r \frac{dr}{dt} \quad \rightarrow \quad \frac{dr}{dt} = \frac{dA}{dt} \cdot \frac{1}{2\pi r} \end{aligned}$$

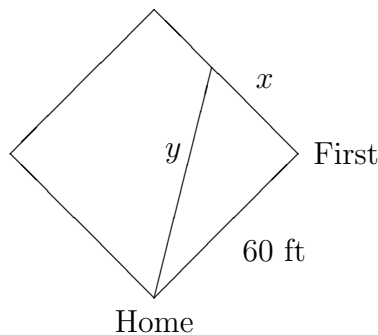
Now find we are able to find $dr/dt|_{A=9}$.

$$\frac{dr}{dt} \Big|_{A=9} = \frac{dA}{dt} \Big|_{A=9} \cdot \frac{1}{2\pi \frac{3}{\sqrt{\pi}}} = 6 \cdot \frac{1}{\frac{6\pi}{\sqrt{\pi}}} = 6 \cdot \frac{\sqrt{\pi}}{6\pi} = \frac{\sqrt{\pi}}{\pi} = \boxed{\frac{1}{\sqrt{\pi}} \text{ mi/h}}$$

7. A softball diamond is a square whose sides are 60 ft long. Suppose that a player running from first to second base has a speed of 25 ft/s at the instant when she is 10 ft from second base. At what rate is the player's distance from home plate changing at that instant?

Let x denote the distance from first base and y the distance from home (See figure). Then we have the following relationship, $60^2 + x^2 = y^2$. Now 10 feet from second base translates to 50 ft from first base. So when $x = 50$,

$$\begin{aligned} 60^2 + x^2 &= y^2 \\ 60^2 + 50^2 &= y^2 \\ \sqrt{3600 + 2500} &= y \\ \sqrt{6100} &= y \\ 10\sqrt{61} &= y \end{aligned}$$



Now find a relationship between dx/dt and dy/dt :

$$\begin{aligned} 60^2 + x^2 &= y^2 \\ \frac{d}{dt}(60^2 + x^2) &= \frac{d}{dt}(y^2) \\ 2x \frac{dx}{dt} &= 2y \frac{dy}{dt} \quad \rightarrow \quad \frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt} \end{aligned}$$

Now we can find dy/dt using the fact that when $x = 50$ we know $y = 10\sqrt{61}$ and $dx/dt = 25 \text{ ft/sec}$.

$$\begin{aligned} \frac{dy}{dt} &= \frac{x}{y} \cdot \frac{dx}{dt} \\ &= \frac{50}{10\sqrt{61}}(25 \text{ ft/s}) \\ &= \boxed{\frac{125}{\sqrt{61}} \text{ ft/s}} \end{aligned}$$

8. A conical water tank with vertex down has a radius of 10 ft at the top and is 24 ft high. If water flows into the tank at a rate of $20 \text{ ft}^3/\text{min}$, how fast is the depth of the water increasing when the water is 16 ft deep?

We are asked to find $\left. \frac{dh}{dt} \right|_{h=16}$ given that $\frac{dV}{dt} = 20$. The volume of the water tank (right circular cone) is given by $V = \frac{1}{3}\pi r^2 h$.

Look at the accompanying figure. We can use similar triangles to obtain a relationship between r and h . Then solve for r and substitute into our volume equation so that volume is a function of height only. Then find dV/dt .

$$\frac{10}{24} = \frac{r}{h}$$

$$\frac{5}{12} \cdot h = r$$

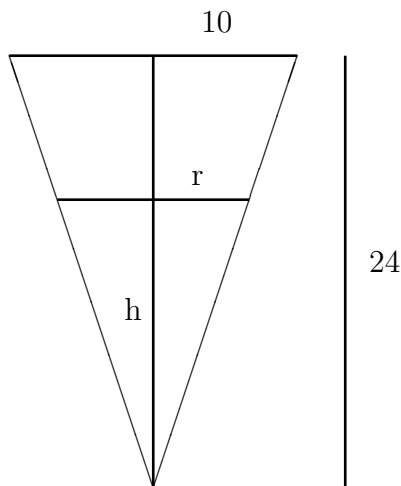
$$V = \frac{\pi}{3} r^2 h$$

$$= \frac{\pi}{3} \left(\frac{5}{12} h \right)^2 h$$

$$= \frac{\pi}{3} \frac{25}{144} h^3$$

$$\frac{dV}{dt} = \frac{d}{dt} \left(\frac{\pi}{3} \frac{25}{144} h^3 \right)$$

$$= \pi h^2 \frac{25}{144} \frac{dh}{dt}$$



Now when $h = 16$:

$$20 = \pi(16)^2 \frac{25}{144} \frac{dh}{dt}$$

$$\rightarrow \frac{dh}{dt} = 20 \frac{1}{16^2 \pi} \frac{144}{25} = \frac{4(5)(4^2)(3^2)}{4^4(5^2)\pi} = \boxed{\frac{9}{20\pi}}$$

9. Grain pouring from a chute at the rate of $8 \text{ ft}^3/\text{min}$ forms a conical pile whose height is always twice its radius. How fast is the height of the pile increasing at the instant when the pile is 6 ft high?

We need to find $dh/dt|_{h=6ft}$, so we need to express the volume of the conical pile, $V = \frac{1}{3}\pi r^2 h$, solely in terms of h . We know that $h = 2r$ so that implies $r = \frac{1}{2}h$. Now substitute r into the volume equation and find dV/dt so we can solve for dh/dt :

$$\begin{aligned} V &= \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 h \\ &= \frac{1}{3}\pi \frac{1}{4}h^2 \cdot h \\ &= \frac{\pi}{12}h^3 \end{aligned}$$

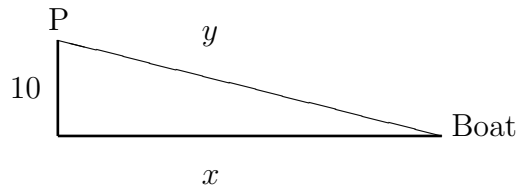
$$\frac{dV}{dt} = 3 \frac{\pi}{12} h^2 \frac{dh}{dt} = \frac{\pi h^2}{4} \frac{dh}{dt}$$

$$\rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$$

$$\left. \frac{dh}{dt} \right|_{h=6ft} = \frac{4}{\pi(6ft)^2} \left. \frac{dV}{dt} \right|_{h=6} = \frac{4}{\pi \cdot 36ft^2} \cdot 8 \frac{ft^3}{min} = \boxed{\frac{8}{9\pi} ft/min}$$

10. A boat is pulled into a dock by means of a rope attached to a pulley on the dock. The rope is attached to the bow of the boat at a point 10 ft below the pulley. If the rope is pulled through the pulley at a rate of 20 ft/min, at what rate will the boat be approaching the dock when 125 ft of rope is out?

Let $x = \text{Distance from dock to the boat}$, and $y = \text{Length of rope}$. Then we have the following relationship: $x^2 + 10^2 = y^2$.



Since the Boat is coming towards the pulley, x and y will shrink. So in this viewpoint, $dy/dt = -20 \text{ ft/min}$. We need to find $dx/dt|_{y=125}$.

Now when $y = 125$ we can find x :

$$\begin{aligned} x^2 + 10^2 &= 125^2 \\ x^2 &= 15625 - 100 = 15525 \\ x &= \sqrt{15525} = \sqrt{15^2 \cdot 69} = 15\sqrt{69} \end{aligned}$$

Now find d/dt of $x^2 + 10^2 = y^2$ to find dx/dt :

$$\begin{aligned} \frac{d}{dt}(x^2 + 10^2) &= \frac{d}{dt}(y^2) \\ 2x \frac{dx}{dt} &= 2y \frac{dy}{dt} \end{aligned}$$

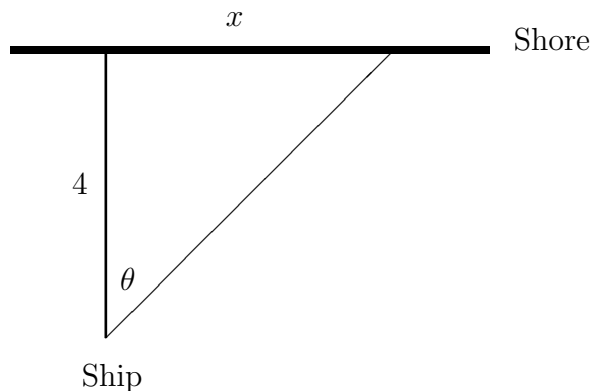
$$\frac{dx}{dt} = \frac{y}{x} \frac{dy}{dt}$$

$$\left. \frac{dx}{dt} \right|_{y=125} = \frac{125}{15\sqrt{69}} \cdot \left. \frac{dy}{dt} \right|_{y=125} = \frac{25}{3\sqrt{69}}(-20) = -\frac{500}{3\sqrt{69}}$$

The boat approaches the dock at $+\frac{500}{3\sqrt{69}} \text{ ft/min}$ (Why do we not use the negative (-) solution for dx/dt ? We setup the problem so that dx/dt is the rate at which the boat is coming towards the pulley, so it makes sense to say it has a positive rate, even though the distance is shrinking.)

11. A beacon that makes one revolution every 10 seconds is located on a ship anchored 4 kilometers from a straight shoreline. How fast is the beam moving along the shoreline when it makes an angle of 45° with the shore?

Look at the picture below. We are asked to find $\frac{dx}{dt}$ when $\theta = 45^\circ = \frac{\pi}{4}$. We are given that $\frac{d\theta}{dt} = 10$ revolutions per minute. Since there are 2π radians in a complete revolution, we have $\frac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5}$ rad/sec.



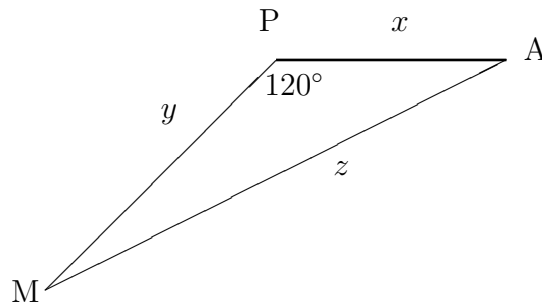
We need a relationship then between θ and x . We see that $\tan(\theta) = \frac{x}{4}$ so set $x = 4 \tan(\theta)$ and find dx/dt :

$$\begin{aligned}
 x &= 4 \tan(\theta) \\
 \frac{d}{dt}(x) &= \frac{d}{dt}(4 \tan(\theta)) \\
 \frac{dx}{dt} &= 4 \sec^2(\theta) \frac{d\theta}{dt}
 \end{aligned}$$

$$\left. \frac{dx}{dt} \right|_{\theta=\pi/4} = 4 \sec^2(\pi/4) \left. \frac{d\theta}{dt} \right|_{\theta=\pi/4} = 4 (\sqrt{2})^2 \frac{\pi}{5} = \boxed{\frac{8\pi}{5} \text{ km/s}}$$

12. An aircraft is flying at a constant altitude with a constant speed of 600 mi/h. An antiaircraft missile is fired on a straight line at an angle 120° to the flight path of the aircraft so that it will hit the aircraft at a point P. At the instant the aircraft is 2 mi from the impact point P, the missile is 4 mi from P and flying at 1200 mi/h. At that instant, how rapidly is the distance between the missile and the aircraft decreasing?

Let x = distance from P to aircraft (A), y = distance P to missile (M), and z = distance between missile and aircraft (See figure below). We are asked to find $\frac{dz}{dt}$ when $x = 2$, $y = 4$ given that $\frac{dx}{dt} = -600$ and $\frac{dy}{dt} = -1200$ at his instant (They are negative because they are shrinking).



Using the law of cosines, and that $120^\circ = \frac{2\pi}{3}$, we have a relationship between x , y , z :

$$\begin{aligned} z^2 &= x^2 + y^2 - 2xy \cos\left(\frac{2\pi}{3}\right) \\ &= x^2 + y^2 - 2xy \left(-\frac{1}{2}\right) \end{aligned}$$

$$\rightarrow z^2 = x^2 + y^2 + xy \quad (*)$$

Now when we need to find z when $x = 2$, $y = 4$:

$$\begin{aligned} z^2 &= (2)^2 + (4)^2 + 2(4) \\ &= 4 + 16 + 8 \\ &= 28 \\ \rightarrow z &= \sqrt{28} = 2\sqrt{7} \end{aligned}$$

Now take take d/dt of (*) and solve for dz/dt :

$$\frac{d}{dt}(z^2) = \frac{d}{dt}(x^2 + y^2 + xy)$$

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + x \frac{dy}{dt} + y \frac{dx}{dt}$$

$$\rightarrow \frac{dz}{dt} = \frac{1}{2z} \left((2x + y) \frac{dx}{dt} + (2y + x) \frac{dy}{dt} \right)$$

Now when $x = 2$, $y = 4 \dots$

$$\begin{aligned} \left. \frac{dz}{dt} \right|_{y=4}^{x=2} &= \frac{1}{2(2\sqrt{7})} \left((2(2) + 4) \left. \frac{dx}{dt} \right|_{y=4}^{x=2} + (2(4) + 2) \left. \frac{dy}{dt} \right|_{y=4}^{x=2} \right) \\ &= \frac{1}{4\sqrt{7}} (8(-600) + 10(-1200)) \\ &= \frac{1}{4\sqrt{7}} (-4800 - 12000) \\ &= \frac{1}{4\sqrt{7}} (-16800) \\ &= -\frac{4200}{\sqrt{7}} = -\frac{4200\sqrt{7}}{7} = \boxed{-600\sqrt{7} \text{ mi/h}} \end{aligned}$$

This means that the distance between missile and the aircraft is decreasing at the rate of $600\sqrt{7}$ miles per hour.

13. A point P is moving along the curve whose equation is $y = 2x$. How fast is the distance between P and the point $(3, 0)$ changing at the instant when P is $(3, 6)$ if x is decreasing at the rate of 2 units/sec at that instant?

For any x , the coordinates of P is given by $(x, 2x)$. The distance between $P = (x, 2x)$ and $(3, 0)$ is given by the distance formula:

$$\begin{aligned} D &= \sqrt{(x-3)^2 + (2x-0)^2} \\ &= \sqrt{x^2 - 6x + 9 + 4x^2} \\ &= \sqrt{5x^2 - 6x + 9} \end{aligned}$$

Now we need to find $dD/dt|_{x=3}$ with the fact that $dx/dt|_{x=3} = -2$:

$$\begin{aligned} \frac{dD}{dt} &= \frac{1}{2}(5x^2 - 6x + 9)^{-1/2} \left(10x \frac{dx}{dt} - 6 \frac{dx}{dt} \right) \\ &= \frac{(10x - 6) \frac{dx}{dt}}{2\sqrt{5x^2 - 6x + 9}} \\ &= \frac{2(5x - 3)}{2\sqrt{5x^2 - 6x + 9}} \frac{dx}{dt} \\ &= \frac{5x - 3}{\sqrt{5x^2 - 6x + 9}} \frac{dx}{dt} \end{aligned}$$

$$\begin{aligned} \left. \frac{dD}{dt} \right|_{x=3} &= \frac{5(3) - 3}{\sqrt{5(3)^2 - 6(3) + 9}} \left. \frac{dx}{dt} \right|_{x=3} \\ &= \frac{12}{\sqrt{45 - 18 + 9}} (-2) \\ &= -\frac{24}{\sqrt{36}} \\ &= -\frac{24}{6} = \boxed{-4 \text{ units/sec}} \end{aligned}$$

14. A particle is moving along the curve $y = \frac{x}{x^2+1}$. Find all values of x at which the rate of change of x with respect to time is three times that of y . (Assume that dx/dt is never zero.)

We need to solve the equation $\frac{dx}{dt} = 3\frac{dy}{dt}$. Since dx/dt is never zero, we can divide by it. This means that $1 = 3\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, or $\frac{1}{3} = \frac{dy}{dx}$. We will use this later.

Next rewrite the equation as $(x^2 + 1)y = x$ and differentiate with respect to x .

$$\begin{aligned} \frac{d}{dx}((x^2 + 1)y) &= \frac{d}{dx}(x) \\ (x^2 + 1)\frac{dy}{dx} + y(2x) &= 1 && \text{Replace } dy/dx = 1/3 \\ (x^2 + 1)\left(\frac{1}{3}\right) + 2yx &= 1 \\ x^2 + 1 + 6xy &= 3 && \text{Replace } y \text{ with original equation} \\ x^2 + 1 + 6x\left(\frac{x}{x^2 + 1}\right) &= 3 && \text{Multiply both sides by } x^2 + 1 \\ (x^2 + 1)^2 + 6x^2 &= 3(x^2 + 1) \\ x^4 + 2x^2 + 1 + 6x^2 - 3x^2 - 3 &= 0 \\ x^4 + 5x^2 - 2 &= 0 && \text{Substitute } u = x^2 \\ u^2 + 5u - 2 &= 0 && \text{Complete the square} \\ u^2 + 5u + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 - 2 &= 0 \\ \left(u + \frac{5}{2}\right)^2 - \frac{25}{4} - \frac{8}{4} &= 0 \\ \left(u + \frac{5}{2}\right)^2 &= \frac{33}{4} \\ u + \frac{5}{2} &= \pm \frac{\sqrt{33}}{2} \\ u &= -\frac{5}{2} \pm \frac{\sqrt{33}}{2} && \text{Replace } u = x^2 \\ \rightarrow x^2 &= -\frac{5}{2} \pm \frac{\sqrt{33}}{2} \end{aligned}$$

Note that $-\frac{5}{2} - \frac{\sqrt{33}}{2} < 0$, so we must disregard it before taking the square root of a negative number.

$$\rightarrow x^2 = -\frac{5}{2} + \frac{\sqrt{33}}{2} = \frac{-5 + \sqrt{33}}{2}$$

$$x = \pm \sqrt{\frac{-5 + \sqrt{33}}{2}}$$