Local Linear Approximation; Differentials
Solutions To Selected Problems
Calculus 9^{th} Edition Anton, Bivens, Davis

Matthew Staley

October 2, 2011

1. Confirm that the stated formula is the local linear approximation at $x_0 = 0$.

   (a) $(1+x)^{15} \approx 1 + 15x$

   Let $f(x) = (1+x)^{15}$, then $f'(x) = 15(1+x)^{14}\frac{d}{dx}(x) = 15(1+x)^{14}$.
   Now use the local linear approximation formula for $f(x)$:

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) \\ &= (1+x_0)^{15} + 15(1+x_0)^{14}(x - x_0) \\ &= (1+0)^{15} + 15(1+0)^{14}(x-0) \\ &= (1)^{15} + 15(1)^{14}(x) \\ &= \boxed{1 + 15x} \end{aligned}$$

   (b) $\tan(x) \approx x$

   Let $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$.

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) \\ &= \tan(x_0) + \sec^2(x_0)(x - x_0) \\ &= \tan(0) + \sec^2(0)(x - 0) \\ &= 0 + 1^2(x) \\ &= \boxed{x} \end{aligned}$$

   (c) $\frac{1}{1+x} \approx 1 - x$

   Let $f(x) = \frac{1}{1+x} = (1+x)^{-1}$, then $f'(x) = -1(1+x)^{-2}(1) = -\frac{1}{(1+x)^2}$

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) \\ &= \frac{1}{1+x_0} - \frac{1}{(1+x)^2}(x - x_0) \\ &= \frac{1}{1+0} - \frac{1}{(1+0)^2}(x - 0) \\ &= \boxed{1 - x} \end{aligned}$$

2. Confirm that the stated formula is the local linear approximation of f at $x_0 = 1$, where $\triangle x = x - 1$

   (a) $f(x) = x^4$; $(1 + \triangle x)^4 \approx 1 + 4 \triangle x$

   Use the local linear approximation on $f(x) = x^4,\ f'(x) = 4x^3,\ x_0 = 1$

$$\begin{aligned} f(x) &\approx f(1) + f'(1)(x-1) \\ x^4 &\approx (1)^4 + 4(1)^3(x-1) \\ &= 1 + 4(x-1) \end{aligned}$$

   Set $\triangle x = x - 1$; then $x = 1 + \triangle x$.

$$\rightarrow \boxed{(1 + \triangle x)^4 \approx 4 \triangle x}$$

   (b) $f(x) = \frac{1}{2+x}$; $\frac{1}{3+\triangle x} \approx \frac{1}{3} - \frac{1}{9} \triangle x.$

$$\begin{aligned} f(x) &= \frac{1}{2+x} = (2+x)^{-1} \\ f'(x) &= -1(2+x)^{-2}(1) \\ &= -\frac{1}{(2+x)^2} \end{aligned}$$

   Use the local linear approximation on $f(x),\ f'(x),\ x_0 = 1$

$$\begin{aligned} f(x) &\approx f(1) + f'(1)(x-1) \\ \frac{1}{2+x} &\approx \frac{1}{2+1} - \frac{1}{(2+1)^2}(x-1) \\ &= \frac{1}{3} - \frac{1}{9}(x-1) \end{aligned}$$

   Since $\triangle x = x - 1$, we can add 3 to both sides to obtain $3 + \triangle x = x + 2$.

$$\rightarrow \frac{1}{2+x} = \boxed{\frac{1}{3+\triangle x} \approx \frac{1}{3} - \frac{1}{9} \triangle x}$$

3. Use an appropriate local linear approximation to estimate the value of the given quantity.

(a) $(3.02)^4$

By inspection we see that can be $f(x) = (x)^4$. Then $x_0 = 3$ and $\triangle x = 0.02$, with $f'(x) = 4x^3$.

$$
\begin{aligned}
f(3 + \triangle x) &\approx f(3) + f'(3) \triangle x \\
f(3.02) &\approx (3)^4 + 4(3)^3(0.02) \\
&= 81 + 108\frac{2}{100} \\
&= 81 + 108\frac{1}{50} \\
&= 81 + \frac{54}{25} \\
&= 81 + 2\frac{4}{25} \\
&= \boxed{83\frac{4}{25} = 83.16}
\end{aligned}
$$

$(3.02)^4 \approx 83.16$ with the actual value to 4 decimals places of 83.181696.

(b) $\sqrt{65}$

The closest perfect square to this is $\sqrt{64} = 8$. So $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, $x_0 = 64$ and $\triangle x = 1$.

$$
\begin{aligned}
f(64 + \triangle x) &= f(64) + f'(64) \triangle x \\
\sqrt{65} &\approx \sqrt{64} + \frac{1}{2\sqrt{64}}(1) \\
&= 8 + \frac{1}{16} \\
&= \boxed{8\frac{1}{16} = 8.0625}
\end{aligned}
$$

$\sqrt{65} \approx 8.0625$ with the actual value to 5 decimal places of 8.06225.

(c) $\sin(0.1)$

$f(x) = \sin(x),\ f'(x) = \cos(x),\ x_0 = 0$, and $\triangle x = 0.1$.

$$\begin{aligned} f(0 + \triangle x) &\approx f(0) + f'(0) \triangle x \\ \sin(0.1) &\approx \sin(0) + \cos(0)(0.1) \\ &= 0 + 1(0.1) \\ &= \boxed{0.1} \end{aligned}$$

Actual value to 3 decimal places is 0.099.

(d) $\tan(0.2)$

$f(x) = \tan(x),\ f'(x) = \sec^2(x),\ x_0 = 0$, and $\triangle x = 0.2$.

$$\begin{aligned} f(0 + \triangle x) &\approx f(0) + f'(0) \triangle x \\ \tan(0.2) &\approx \tan(0) + \sec^2(0)(0.2) \\ &= 0 + 1^2(0.2) \\ &= \boxed{0.2} \end{aligned}$$

Actual value to 3 decimal places is 0.202.

(e) $\cos(31^\circ)$

31° is close to $30^\circ = \pi/6$, so choose $f(x) = \cos(x),\ f'(x) = -\sin(x)$, $x_0 = \pi/6$ and $\triangle x = 1^\circ = \pi/180$.

$$\begin{aligned} f(30^\circ + \triangle x) &\approx f(\pi/6) + f'(\pi/6) \triangle x \\ \cos(31^\circ) &\approx \cos(\pi/6) - \sin(\pi/6)(\pi/180) \\ &= (\sqrt{3}/2) - (1/2)(\pi/180) \\ &= \boxed{\frac{\sqrt{3}}{2} - \frac{\pi}{360} \approx 0.8573} \end{aligned}$$

Actual value to 5 decimal places is 0.85716.

4. Find formulas for dy and $\triangle y$.

(a) $$y = x^2 - 2x + 1$$

$$\frac{dy}{dx} = 2x - 2$$

$$\boxed{dy = (2x - 2)dx}$$

$$\begin{aligned}\triangle y &= f(x + \triangle x) - f(x) \\ &= [(x + \triangle x)^2 - 2(x + \triangle x) + 1] - [x^2 - 2x + 1] \\ &= x^2 + 2x \triangle x + (\triangle x)^2 - 2x - 2 \triangle x + 1 - x^2 + 2x - 1 \\ &= \boxed{2x \triangle x + (\triangle x)^2 - 2 \triangle x}\end{aligned}$$

(b) $$y = \sin(x)$$

$$\frac{dy}{dx} = \cos(x)$$

$$\boxed{dy = \cos(x)dx}$$

$$\begin{aligned}\triangle y &= f(x + \triangle x) - f(x) \\ &= \boxed{\sin(x + \triangle x) - \sin(x)}\end{aligned}$$

5. Find the differential dy.

(a)
$$y = 4x^3 - 7x^2$$

$$\begin{aligned} dy &= d(4x^3 - 7x^2) \\ &= 12x^2dx - 14xdx \\ &= \boxed{(12x^2 - 14x)dx} \end{aligned}$$

(b)
$$y = x\cos(x)$$

$$\begin{aligned} dy &= d(x\cos(x)) \\ &= xd\cos(x) + \cos(x)dx \\ &= x(-\sin(x))dx + \cos(x)dx \\ &= \boxed{(-x\sin(x) + \cos(x))dx} \end{aligned}$$

(c)
$$y = \frac{1}{x} = x^{-1}$$

$$\begin{aligned} dy &= d(x^{-1}) \\ &= -1x^{-2}dx \\ &= \boxed{-\frac{1}{x^2}dx} \end{aligned}$$

(d)
$$y = x\sqrt{1-x} = x(1-x)^{1/2}$$

$$\begin{aligned}
dy &= d(x(1-x)^{1/2}) \\
&= x\ d(1-x)^{1/2} + (1-x)^{1/2}\ dx \\
&= x\left(\frac{1}{2}(1-x)^{-1/2}(-1)dx\right) + (1-x)^{1/2}\ dx \\
&= -\frac{x}{2\sqrt{1-x}}dx + \sqrt{1-x}\ dx \\
&= \left(\sqrt{1-x} - \frac{x}{2\sqrt{1-x}}\right)dx \\
&= \left(\sqrt{1-x}\cdot\frac{2\sqrt{1-x}}{2\sqrt{1-x}} - \frac{x}{2\sqrt{1-x}}\right)dx \\
&= \left(\frac{2(1-x)-x}{2\sqrt{x}}\right)dx \\
&= \boxed{\left(\frac{2-3x}{2\sqrt{1-x}}\right)dx}
\end{aligned}$$

(e)
$$y = (1+x)^{-17}$$

$$\begin{aligned}
dy &= d(1+x)^{-17} \\
&= -17(1+x)^{-18}\ dx \\
&= \boxed{-\frac{17}{(1+x)^{18}}dx}
\end{aligned}$$

6. Use the differential dy to approximate $\triangle y$ when x changes as indicated.

(a) $y = \sqrt{3x-2}$; from $x = 2$ to $x = 2.03$, $\to dx = 0.03$

$$\begin{aligned} dy &= \frac{1}{2}(3x-2)^{-1/2}(3)\ dx \\ &= \frac{3}{2\sqrt{3x-2}}dx \end{aligned}$$

$$\begin{aligned} \triangle y \approx dy &= \frac{3}{2\sqrt{3x-2}}dx \\ &= \frac{3}{2\sqrt{3(2)-2}}(0.03) \\ &= \frac{3}{2\sqrt{4}}(0.03) \\ &= \frac{3}{4}\left(\frac{3}{100}\right) \\ &= \boxed{\frac{9}{400} = 0.0225} \end{aligned}$$

(b) $y = \frac{x}{x^2+1}$; from $x = 2$ to $x = 1.96$, $\to\ dx = -0.04$

$$\begin{aligned} dy &= \frac{(x^2+1)dx - x(2xdx)}{(x^2+1)^2} \\ &= \frac{(x^2-2x^2+1)dx}{(x^2+1)^2} \\ &= \frac{1-x^2}{(x^2+1)^2}dx \end{aligned}$$

$$\begin{aligned} \triangle y \approx dy &= \frac{1-2^2}{(2^2+1)^2}(-0.04) \\ &= \frac{1-4}{5^2}\left(-\frac{4}{100}\right) \\ &= -\frac{3}{25}\left(-\frac{1}{25}\right) \\ &= \frac{3}{25^2} = \boxed{\frac{3}{625} = 0.0048} \end{aligned}$$

7. The side of a square is measured to be 10 ft, with a possible error of ± 0.1 ft.

   (a) Use differentials to estimate the error in the calculated area.

   Let x be the side length of the square. Then area $= A = x^2$; $dA = 2xdx$. In this case $x = 10$ and $dx = \pm\, 0.1$. Thus,

$$\begin{aligned} dA &= 2xdx \\ &= 2(10)(\pm\, 0.1) \\ &= \pm\, 20\frac{1}{10} = \pm\, 2ft \end{aligned}$$

   (b) Estimate the percentage errors in the side and the area.

   The relative error in the side, or x, is $\approx \frac{dx}{x} = \frac{\pm\, 0.1}{10} = \pm\, \frac{1}{10}\frac{1}{10} = \pm\, \frac{1}{100} = \pm\, 0.01$. So the percentage error in x is $\approx\ \pm\, 1\%$.

   The relative error in the Area, or A is $\approx \frac{dA}{A} = \frac{2xdx}{x^2} = 2\frac{dx}{x} = 2(\pm\, 0.01) = \pm\, 0.02$. So the percentage error in x is $\approx \pm\, 2\%$

8. The electrical resistance R of a certain wire is given by $R = k/r^2$, where k is a constant and r is the radius of the wire. Assuming that the radius r has a possible error of ± 5%, use differentials to estimate the percentage error in R. (Assume k is exact).

   We are asked to find $\frac{dR}{R}$. So $dR = -2kr^{-3}dr = -\frac{2k}{r^3}dr$. Now find $\frac{dR}{R}$ along with the fact that $\frac{dr}{r} = \pm\, 0.05$.

$$\begin{aligned} \frac{dR}{R} &= -\frac{2k}{r^3}dr\frac{1}{R} \\ &= -\frac{2k}{r^3}dr\frac{r^2}{k} \\ &= -2\frac{dr}{r} \\ &= -2(\pm\, 0.05) \\ &= \pm\, 0.10 \end{aligned}$$

   So the percentage error in R is $\approx\ 10\%$.