# Analysis of Functions: Increase, Decrease, and Concavity
# Solutions To Selected Problems
# Calculus $9^{th}$ Edition Anton, Bivens, Davis

Matthew Staley

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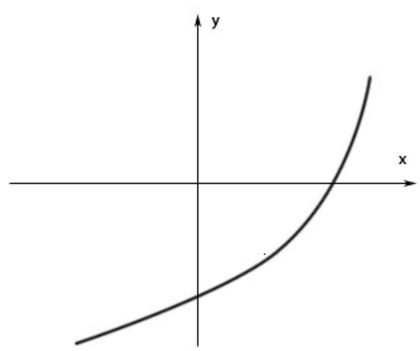
1. In each part, sketch the graph of the function $f$ with the stated properties, and discuss the signs of $f'$ and $f''$.

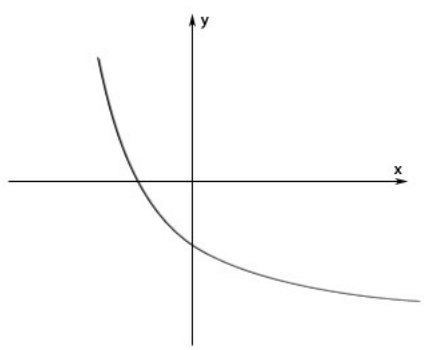
   (a) The function $f$ is concave up and increasing on the interval $(-\infty,\ +\infty)$.

   $f$ increasing means $f' > 0$ and $f$ concave up means $f'' > 0$. There is more than one right way to sketch the graph. Here is one example.



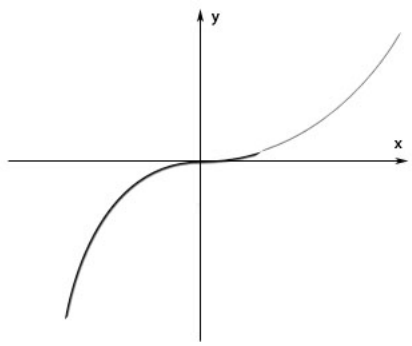

   (b) The function $f$ is concave up and decreasing on the interval $(-\infty,\ +\infty)$.

   $f' < 0$ and $f'' > 0$.

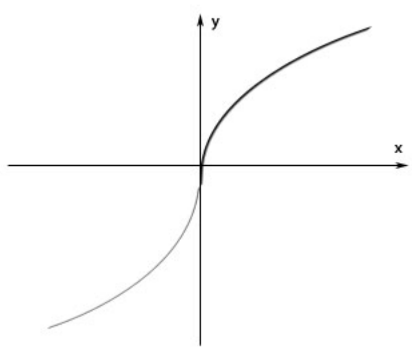


(c) $f$ is increasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave up on $(0, +\infty)$.

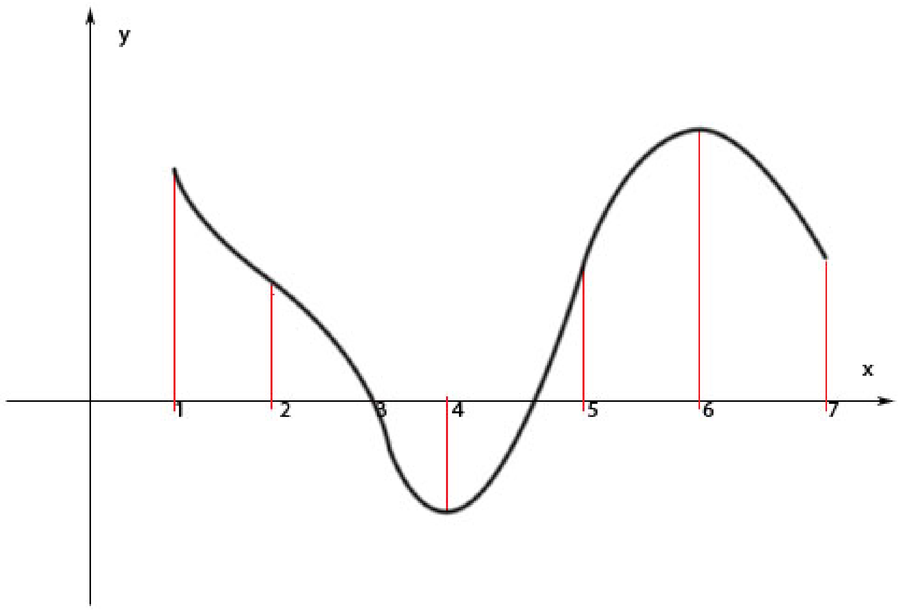
$f' \geq 0$ for all $x$,
$f'' < 0$ on $(-\infty, 0)$
$f'' = 0$ at $x = 0$
$f'' > 0$ on $(0, +\infty)$




(d) $f$ is increasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave down on $(0, \infty)$.



2. In each part, use the graph of $y = f(x)$ below to find the following:




(a) Find the intervals on which $f$ is increasing.
$f$ is increasing wherever $f' > 0$. This happens on the interval $[4, 6]$.

(b) FInd the intervals on which $f$ is decreasing.
$f$ is decreasing wherever $f' < 0$. This happens on $[1, 4]$ and $[6, 7]$.

(c) Find the open intervals on which $f$ is concave up.
By inspection we see that this occurs on $(1, 2)$ and $(3, 5)$.

(d) Find the open intervals on which $f$ is concave down.
By inspection we see that this occurs on $(2, 3)$ and $(5, 7)$.

(e) Find all the values of $x$ at which $f$ has an inflection point.
Inflection points occur when there is a change in concavity. This happens at $x = 2, 3, 5$.

3. A sign chart is presented for the first and second derivatives of a function $f$. Assuming that $f$ is continuous everywhere, find: (a) the intervals on which $f$ is increasing, (b) the intervals on which $f$ is decreasing, (c) the open intervals on which $f$ is concave up, (d) the open intervals on which $f$ is concave down, and (e) the $x$-coordinates of all inflection points.

| Interval | Sign of $f'(x)$ | Sign of $f''(x)$ |
|---|---|---|
| $x < 1$ | $-$ | $+$ |
| $1 < x < 2$ | $+$ | $+$ |
| $2 < x < 3$ | $+$ | $-$ |
| $3 < x < 4$ | $-$ | $-$ |
| $4 < x$ | $-$ | $+$ |

(a) $f$ is increasing on $[1, 3]$.

(b) $f$ is decreasing on $(-\infty, 1]$ and $[3, +\infty)$.

(c) $f$ is concave up on $(-\infty, 2)$ and $(4, +\infty)$.

(d) $f$ is concave down on $(2, 4)$.

(e) Inflection points occur at $x = 3, 4$.

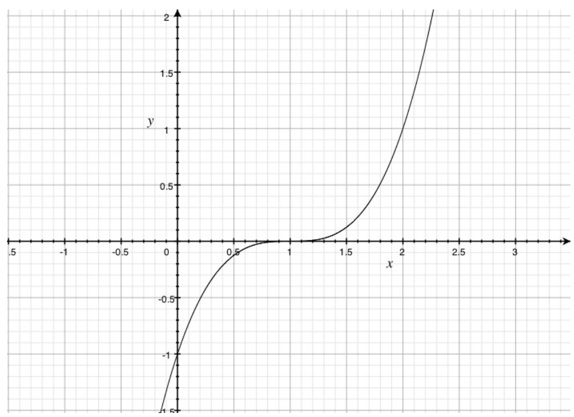
4. True-False:

(a) If $f$ is decreasing on $[0, 2]$, then $f(0) > f(1) > f(2)$.
True - By the definition of a decreasing function.

(b) If $f'(1) > 0$, then $f$ is increasing on $[0, 2]$.
False - There is not enough information. Consider the function $f(x) = \left(x - \frac{1}{2}\right)^2$ with $f'(x) = 2\left(x - \frac{1}{2}\right) = 2x - 1$. Now $f'(1) = 1 > 0$ but $f'(1/4) = 1/2 - 1 = -1/2 < 0$ so $f$ is not increasing on $[0, 2]$.

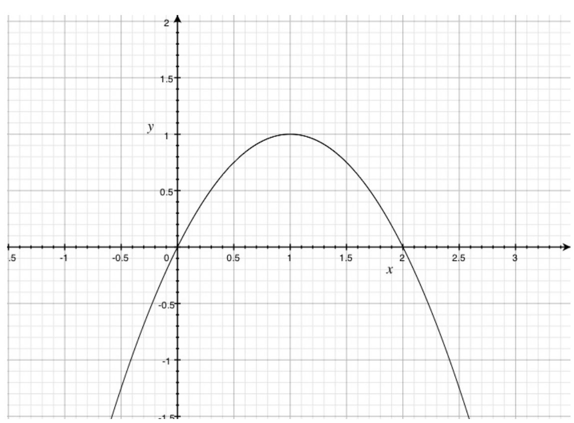
(c) If $f$ is increasing on $[0, 2]$, then $f'(1) > 0$.

False - It is possible for $f'(1) = 0$ and still be increasing. Consider the function $f(x) = (x-1)^3$, so $f'(x) = 3(x-1)^2$ with $f'(1) = 0$ and $f$ increasing on $[0, 2]$. (*see figure below*)



(d) If $f'$ is increasing on $[0, 1]$ and $f'$ is decreasing on $[1, 2]$, then $f$ has an inflection point at $x = 1$.

False - Inflection points only happen when there is a change in concavity, which means a change in the sign of $f''$, not $f'$. As an example, consider $f(x) = -(x-1)^2 + 1$ which is increasing on $[0, 1]$, decreasing on $[1, 2]$, yet always concave down. (*see figure below*)



5. For the following functions find the intervals on which $f$ is increasing, decreasing, concave up, concave down and the points of inflection.

(*We will find the first and second derivates, their possible zeros and then use this information to make a sign chart.*)

(a)

$$f(x) = x^2 - 3x + 8$$

$$\begin{aligned} f'(x) &= 2x - 3 = 0 \\ 2x &= 3 \\ x &= 3/2 \end{aligned}$$

$$f''(x) = 2 > 0$$

| Interval | Sign of $f'(x)$ | Sign of $f''(x)$ |
|---|---|---|
| $x < 3/2$ | $-$ | $+$ |
| $x = 3/2$ | 0 | $+$ |
| $x > 3/2$ | $+$ | $+$ |

i. Increasing on $[3/2, +\infty)$
ii. Decreasing on $(-\infty, 3/2]$
iii. Concave up everywhere, i.e. $(-\infty, +\infty)$
iv. Concave down nowhere
v. No inflection points

(b)

$$f(x) = (2x+1)^3$$

$$\begin{aligned} f'(x) &= 3(2x+1)^2(2) = 0 \\ 6(2x+1)^2 &= 0 \\ (2x+1)^2 &= 0 \\ 2x+1 &= 0 \\ 2x &= -1 \\ x &= -1/2 \end{aligned}$$

$$\begin{aligned} f''(x) &= 12(2x+1) = 0 \\ 2x+1 &= 0 \\ 2x &= -1 \\ x &= -1/2 \end{aligned}$$

| Interval | Sign of $f'(x)$ | Sign of $f''(x)$ |
|---|---|---|
| $x < -1/2$ | $+$ | $-$ |
| $x = -1/2$ | 0 | 0 |
| $x > -1/2$ | $+$ | $+$ |

i. Increasing on $(-\infty, +\infty)$
ii. Decreasing nowhere
iii. Concave up on $(-1/2, +\infty)$
iv. Concave down on $(-\infty, -1/2)$
v. Inflection point at $x = -1/2$

(c)
$$f(x) = \frac{x-2}{(x^2-x+1)^2}$$

$$f'(x) = \frac{(x^2-x+1)^2(1)-(x-2)(2(x^2-x+1)(2x-1))}{(x^2-x+1)^4}$$

$$= \frac{(x^2-x+1)((x^2-x+1)-2(x-2)(2x-1))}{(x^2-x+1)^4}$$

$$= \frac{x^2-x+1-2(2x^2-5x+2)}{(x^2-x+1)^3}$$

$$= \frac{x^2-x+1-4x^2+10x-4}{(x^2-x+1)^3}$$

$$= \frac{-3x^2+9x-3}{(x^2-x+1)^3}$$

$$= \frac{-3(x^2-3x+1)}{(x^2-x+1)^3}$$

Now $f'(x) = 0$ only when the numerator is equal to zero:

$$-3(x^2-3x+1) = 0$$
$$x^2-3x+1 = 0 \qquad \text{Use the quadratic formula}$$

$$x = \frac{3 \pm \sqrt{(-3)^2-4(1)(1)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{9-4}}{2}$$

$$= \frac{3 \pm \sqrt{5}}{2}$$

So $f'(x) = 0$ when $x = \frac{3+\sqrt{5}}{2}$ and $x = \frac{3-\sqrt{5}}{2}$

Now since $\frac{3+\sqrt{5}}{2} \approx 2.62$, we can test the sign of $f'(x)$ using $x = 2$ and $x = 3$:

$$f'(2) = \frac{-3(2^2 - 3(2) + 1)}{(2^2 - 2 + 1)^3}$$

$$= \frac{-3(4 - 6 + 1)}{3^3}$$

$$= \frac{-(-1)}{3^2} = \frac{1}{9} > 0$$

$$f'(3) = \frac{-3(3^2 - 3(3) + 1)}{(3^2 - 3 + 1)^3}$$

$$= \frac{-3}{7^3} < 0$$

Similarly, $\frac{3-\sqrt{5}}{2} \approx 0.38$, we can use $x = 0$ and $x = 1$ to test the sign of $f'(x)$ around this:

$$f'(0) = \frac{-3(0 - 0 + 1)}{(0 - 0 + 1)^3}$$

$$= -3 < 0$$

$$f'(1) = \frac{-3(1 - 3 + 1)}{(1 - 1 + 1)^3}$$

$$= \frac{-3(-1)}{1} = 3 > 0$$

So here is a sign chart of $f'(x)$:

| Interval | Sign of $f'(x)$ |
|---|---|
| $x < \frac{3-\sqrt{5}}{2}$ | $-$ |
| $\frac{3-\sqrt{5}}{2} < x < \frac{x+\sqrt{5}}{2}$ | $+$ |
| $x > \frac{3+\sqrt{5}}{2}$ | $-$ |

Now we need to find $f''(x)$ and check concavity.

$$f'(x) = \frac{-3(x^2-3x+1)}{(x^2-x+1)^3}$$

$$f''(x) = \frac{(x^2-x+1)^3[-3(2x-3)] - (-3(x^2-3x+1)[3(x^2-x+1)^2(2x-1)])}{(x^2-x+1)^6}$$

$$= \frac{(x^2-x+1)^2[-3(x^2-x+1)(2x-3) + 9(x^2-3x+1)(2x-1)}{(x^2-x+1)^6}$$

$$= \frac{-3(2x^3-5x^2+5x-3) + 9(2x^3-7x^2+5x-1)}{(x^2-x+1)^4}$$

$$= \frac{x^3(-6+18) + x^2(15-63) + x(-15+45) + 9 - 9}{(x^2-x+1)^4}$$

$$= \frac{12x^3-48x^2+30x}{(x^2-x+1)^4}$$

$$= \frac{6x(2x^2-8x+5)}{(x^2-x+1)^4}$$

Now $f''(x) = 0$ only when the numerator equals zero:

$$\begin{aligned} 6x(2x^2 - 8x + 5) &= 0 \quad \text{So } x = 0 \text{ is one solution} \\ 2x^2 - 8x + 5 &= 0 \quad \text{Use quadratic formula} \end{aligned}$$

$$\begin{aligned} x &= \frac{8 \pm \sqrt{(-8)^2 - 4(2)(5)}}{2(2)} \\ &= \frac{8 \pm \sqrt{64 - 40}}{4} \\ &= \frac{8 \pm \sqrt{24}}{4} \\ &= \frac{8 \pm 2\sqrt{6}}{4} \\ &= 2 \pm \frac{\sqrt{6}}{2} \end{aligned}$$

So $f''(x) = 0$ when $x = 0$, $x = 2 + \frac{\sqrt{6}}{2} \approx 3.22$ and $x = 2 - \frac{\sqrt{6}}{2} \approx 0.78$. We can use $x = -1,\ 1/2,\ 1 \text{ (or 3)},\ 4$ to test the sign of $f''(x)$.

$$\begin{aligned} f''(-1) &= \frac{6(-1)(2(-1)^2 - 8(-1) + 5)}{((-1)^2 - (-1) + 1)^4} \\ &= \frac{-6(2 + 8 + 5)}{(1 + 1 + 1)^4} = \frac{-6(15)}{3^4} < 0 \end{aligned}$$

$$f''(1/2) = \frac{6(1/2)(2(1/2)^2 - 8(1/2) + 5)}{((1/2)^2 - 1/2 + 1)^4}$$

$$= \frac{3(2(1/4) - 4 + 5)}{(1/4 + 1/2)^4}$$

$$= \frac{3(1/4 + 1)}{(3/4)^4} > 0$$

$$f''(1) = \frac{6(2 - 8 + 5)}{(1 - 1 + 1)^4}$$

$$= \frac{6(-1)}{1^4} = -6 < 0$$

$$f''(4) = \frac{6(4)(2(4)^2 - 8(4) + 5)}{((4)^2 - 4 + 1)^4}$$

$$= \frac{24(32 - 32 + 5)}{(16 - 4 + 1)^4}$$

$$= \frac{24(5)}{13^4} > 0$$

*(Notice that the denominator was always $> 0$ due to the even degree exponent, so we could of just determined the sign of the numerator, but its good practice anyway...)*

Now since we had different roots for $f'(x)$ and $f''(x)$, we need to place the roots in order from least to greatest in order to help us generate an appropriate sign chart. Looking back at the different values we see that:

$0 < \frac{3-\sqrt{5}}{2} < 2 - \frac{\sqrt{6}}{2} < \frac{3+\sqrt{5}}{2} < 2 + \frac{\sqrt{6}}{2}$

Sign chart for both $f'(x)$ and $f''(x)$:

| Interval | Sign of $f'(x)$ | Sign of $f''(x)$ |
|---|---|---|
| $x < 0$ | $-$ | $-$ |
| $0 < x < \frac{3-\sqrt{5}}{2}$ | $-$ | $+$ |
| $\frac{3-\sqrt{5}}{2} < x < 2 - \frac{\sqrt{6}}{2}$ | $+$ | $+$ |
| $2 - \frac{\sqrt{6}}{2} < x < \frac{3+\sqrt{5}}{2}$ | $+$ | $-$ |
| $\frac{3+\sqrt{5}}{2} < x < 2 + \frac{\sqrt{6}}{2}$ | $-$ | $-$ |
| $x > 2 + \frac{\sqrt{6}}{2}$ | $-$ | $+$ |

i. Increasing on $\left[\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right]$

ii. Decreasing on $\left(-\infty, \frac{3-\sqrt{5}}{2}\right]$, $\left[\frac{3+\sqrt{5}}{2}, +\infty\right)$

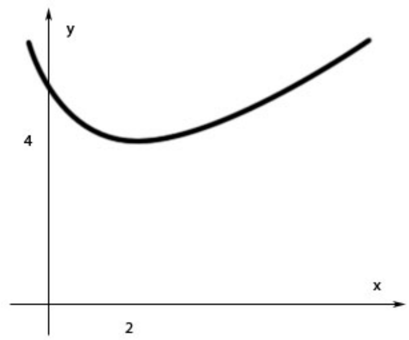
iii. Concave up on $\left(0, 2 - \frac{\sqrt{6}}{2}\right)$, $\left(2 + \frac{\sqrt{6}}{2}, +\infty\right)$

iv. Concave down on $(-\infty, 0)$, $\left(2 - \frac{\sqrt{6}}{2}, 2 + \frac{\sqrt{6}}{2}\right)$

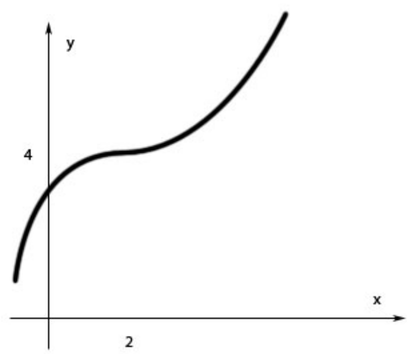
v. Inflection points at $x = 0,\ 2 - \frac{\sqrt{6}}{2},\ 2 + \frac{\sqrt{6}}{2}$

6. In parts (a)-(c), sketch a continuous curve $y = f(x)$ with the stated properties.

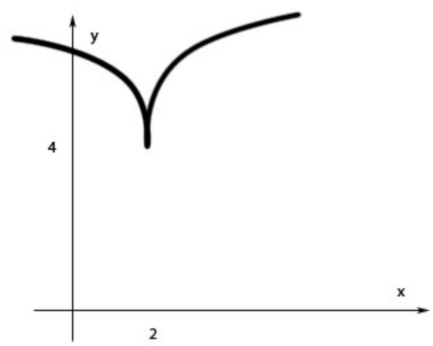
(a) $f(2) = 4$, $f'(2) = 0$, $f''(0) > 0$ for all $x$.




(b) $f(2) = 4$, $f'(2) = 0, f''(x) < 0$ for $x < 2$, $f''(x) > 0$ for $x > 2$.



(c) $f(2) = 4$, $f''(x) > 0$ for $x \neq 2$ and $\lim_{x \to 2^+} f'(x) = +\infty$, $\lim_{x \to 2^-} f'(x) = -\infty$.




7. Use the definition of an increasing function to prove that $f(x) = x^2$ is increasing on $[0, +\infty)$

   Let $0 < x_1 < x_2 < +\infty$. We need to show that $f(x_1) < f(x_2)$ .

   $$f(x_1) - f(x_2) = x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2)$$

   Now clearly $(x_1 + x_2) > 0$ as it is the sum of two positive numbers. What about $(x_1 - x_2)$? Since $x_1 < x_2$, then $x_1 - x_2 < 0$. So $(x_1 - x_2)(x_1 + x_2) < 0$, therefore $f(x_1) - f(x_2) < 0$, or $f(x_1) < f(x_2)$, and thus $f(x) = x^2$ is an increasing function on $[0, +\infty)$.

8. Determine whether the statements are true or false. If a statement is false, find functions for which the statement fails to hold.

   (a) If $f$ and $g$ are increasing on an interval, then so is $f + g$.

   True - Let $x_1 < x_2$ for some $x_1,\ x_2$ in an interval $I$. Then it follows from the definition of an increasing function that $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$. Therefore $f(x_1) + g(x_1) < f(x_2) + g(x_2)$ which is the same as $(f + g)(x_1) < (f + g)(x_2)$, which shows that $f + g$ is also increasing.

   (b) If $f$ and $g$ are increasing on an interval, then so is $f \cdot g$.

   This is a trick question in that there are two cases...

   Case I: True if $f$ and $g$ are both positive, increasing functions, i.e. if $0 < f(x_1) < f(x_2)$ and $0 < g(x_1) < g(x_2)$ for $x_1 < x_2$, where $x_1,\ x_2$ in some interval $I$, then it follows that $f(x_1)g(x_1) < f(x_1)g(x_2)$, or $(f \cdot g)(x_1) < (f \cdot g)(x_2)$, which shows that $f \cdot g$ is increasing.

   Case II: False if $f$ and $g$ are not necessarily positive on the interval $I$. For example, let $f(x) = g(x) = x$ on the interval $(-\infty, 0)$. Note that $f$ and $g$ are still increasing functions on the interval. Now $(f \cdot g)(x) = f(x)g(x) = x \cdot x = x^2$, but we know that this is *decreasing* on $(-\infty, 0)$.