Analysis of Functions: Relative Extrema; Graphing Polynomials
Solutions To Selected Problems
Calculus 9^{th} Edition Anton, Bivens, Davis

Matthew Staley

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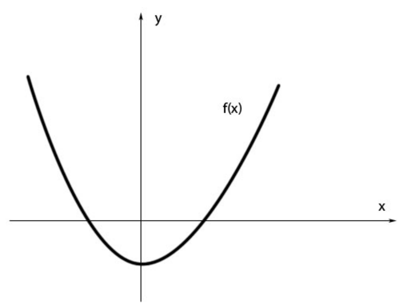
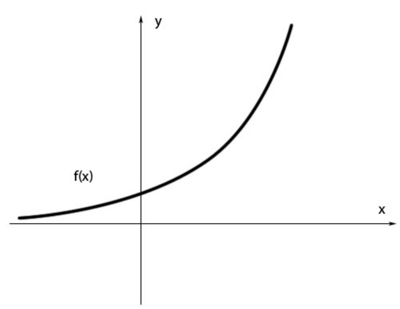
1. In each part, sketch the graph of a continuous function f with the stated properties.

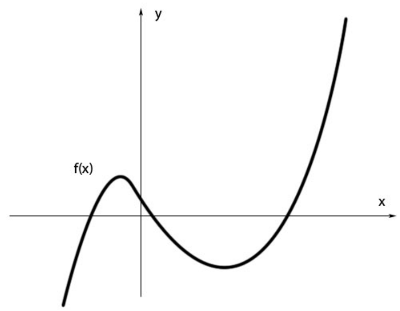
   (a) f is concave up on the interval $(-\infty, +\infty)$ and has exactly one relative extremum.



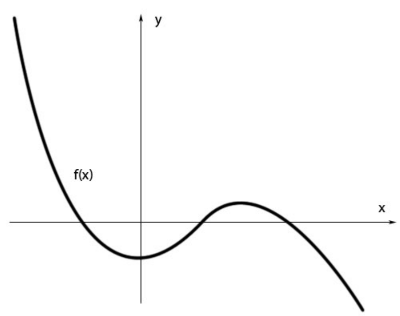
   (b) f is concave up on the interval $(-\infty, +\infty)$ and has no relative extremum.



(c) The function f has exactly two relative extrema on the interval $(-\infty, +\infty)$, and $f(x) \to +\infty$ as $x \to +\infty$.



(d) The function f has exactly two relative extrema on the interval $(-\infty, +\infty)$, and $f(x) \to -\infty$ as $x \to +\infty$.



2. Use the both the first and second derivative tests to show that $f(x) = 3x^2 - 6x + 1$ has a relative minimum at $x = 1$.

$$f(x) = 3x^2 - 6x + 1$$

$$f'(x) = 6x - 6$$

$$f''(x) = 6$$

Now $f'(x) = 0$ when $x = 1$. For the first derivative test, $f'(x) < 0$ for $x < 1$, and $f'(x) > 0$ for $x > 1$. So we have a relative minimum here at $x = 1$.

For the second derivative test, we see that $f''(x) = 6 > 0$ for every x, thus $f''(1) > 0$ as well and again implies that we have a relative minimum at $x = 1$.

3. Use both the first and second derivative tests to show that $f(x) = x^3 - 3x + 3$ has a relative minimum at $x = 1$ and a relative maximum at $x = -1$.

$$f(x) = x^3 - 3x + 3$$

$$\begin{aligned} f'(x) &= 3x^2 - 3 = 0 \\ 3(x^2 - 1) &= 0 \\ x^2 - 1 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

$$\begin{aligned} f''(x) &= 6x = 0 \\ x &= 0 \end{aligned}$$

We see that $f'(x) < 0$ for $-1 < x < 1$, and $f'(x) > 0$ for $x < -1$ and $x > 1$, so by the first derivative test, we have a relative minimum at $x = 1$, and a relative maximum at $x = -1$. By the second derivative test, we have $f'(1) = 0$, $f''(1) > 0$, and $f'(-1) = 0$, $f''(-1) < 0$ and thus a relative minimum at $x = 1$ and a relative maximum at $x = -1$.

4. Show that both of the functions $f(x) = (x-1)^4$ and $g(x) = x^3 - 3x^2 + 3x - 2$ have stationary points at $x = 1$. What does the first and second derivative tests tell you about the nature of those stationary points?

Stationary points occur when the derivative evaluated at that point is zero. So we need to show that $f'(1) = 0$ and $g'(1) = 0$.

$$\begin{aligned} f(x) &= (x-1)^4 \\ f'(x) &= 4(x-1)^3 \\ f'(1) &= 4(1-1)^3 = 0 \end{aligned}$$

$$\begin{aligned} g(x) &= x^3 - 3x^2 + 3x - 2 \\ g'(x) &= 3x^2 - 6x + 3 \\ &= 3(x^2 - 2x + 1) \\ &= 3(x-1)^2 \\ g'(1) &= 3(1-1)^2 = 0 \end{aligned}$$

Checking the first derivative test for f, we see that $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$, so f has a relative minimum at $x = 1$. For g, we see that $g'(x) > 0$ for every x, so g is increasing and there is no relative extremum at $x = 1$.

For the second derivative test, we need to find the second derivatives of f and g

$f''(x) = 12(x-1)^2$ with $f''(1) = 0$ which means that the second derivative test is inconclusive for f.

$g''(x) = 6(x-1)$ with $g''(1) = 0$ which also means that the second derivative test is inconclusive for g.

5. Locate the critical points and identify which critical points are stationary points.

(a)
$$f(x) = 4x^4 - 16x^2 + 17$$

$$\begin{aligned} f'(x) &= 16x^3 - 32x \\ &= 16x(x^2 - 2) \end{aligned}$$

$\boxed{x = 0, \pm\sqrt{2}}$ are the stationary points.

(b)
$$f(x) = \frac{x+1}{x^2+3}$$

$$\begin{aligned} f'(x) &= \frac{(x^2+3)(1) - (x+1)(2x)}{(x^2+3)^2} \\ &= \frac{x^2 + 3 - 2x^2 - 2x}{(x^2+3)^2} \\ &= \frac{-x^2 - 2x + 3}{(x^2+3)^2} \\ &= -\frac{(x + 2x - 3)}{(x^2+3)^2} \\ &= -\frac{(x+3)(x-1)}{(x^2+3)^2} \end{aligned}$$

$\boxed{x = -3, 1}$ are the stationary points.

(c)
$$f(x) = |\sin(x)| = \begin{cases} \sin(x), & \sin(x) \geq 0 \\ -\sin(x), & \sin(x) < 0 \end{cases}$$

$$f'(x) = \begin{cases} \cos(x), & \sin(x) > 0 \\ -\cos(x), & \sin(x) < 0 \end{cases}$$

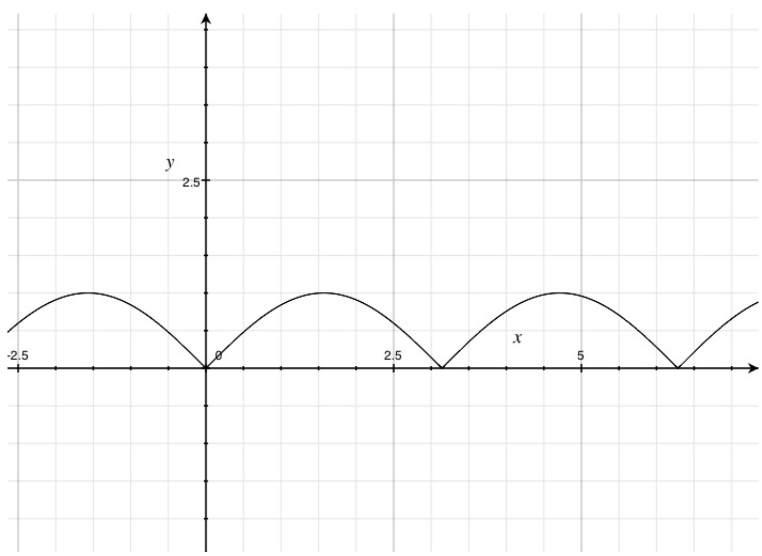
Now $f'(x) = 0$ wherever $\pm\cos(x) = 0$ which is at $x = \frac{\pi}{2} + n\pi$,
$n = 0,\ \pm 1,\ \pm 2 \ldots$

We must understand that there exist critical points x where f is *not* differentiable. Where? $f'(x)$ does not exist on the points where $\sin(x) = 0$, so $x = n\pi$,
$n = 0,\ \pm 1,\ \pm 2,\ \ldots$ as

$\lim_{x \to n\pi^-}\ f'(x) \to\ \cos(n\pi) = -1$ and
$\lim_{x \to n\pi^+}\ f'(x) \to\ -\cos(n\pi) = +1$

The critical points are $\dfrac{\pi}{2}n$. The stationary points are $\dfrac{\pi}{2} + n\pi$ for every integer n

Here is a graph of $f(x) = |\sin(x)|$ to help you visualize the problem.



6. True-False: Assume that f is continuous everywhere.

(a) If f has a relative max at $x = 1$, then $f(1) \geq f(2)$.

False - The relative max at $x = 1$ implies nothing about $f(2)$.

(b) If f has a relative max at $x = 1$, then $x = 1$ is a critical point for f.

True- This is the theorem.

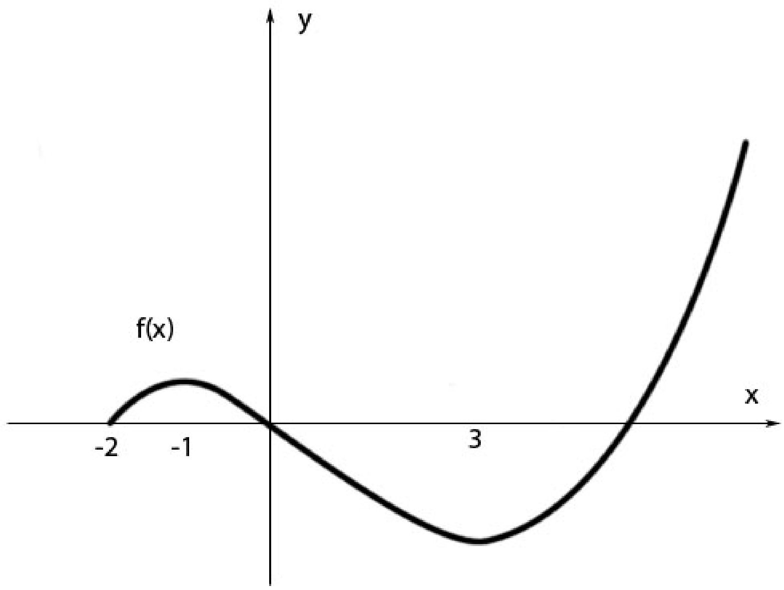
(c) If $f''(x) > 0$, then f has a relative minimum at $x = 1$.

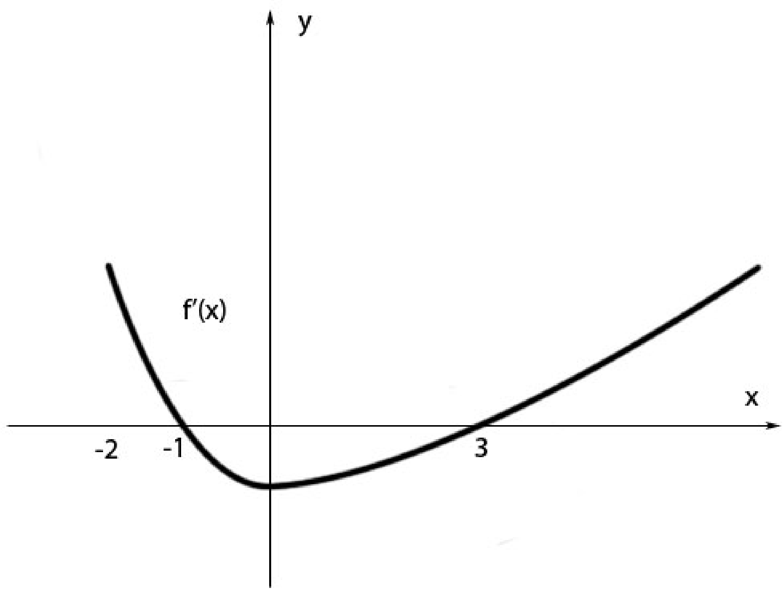
False- Not enough information to use the second derivative test. We would need to have $f'(1) = 0$.

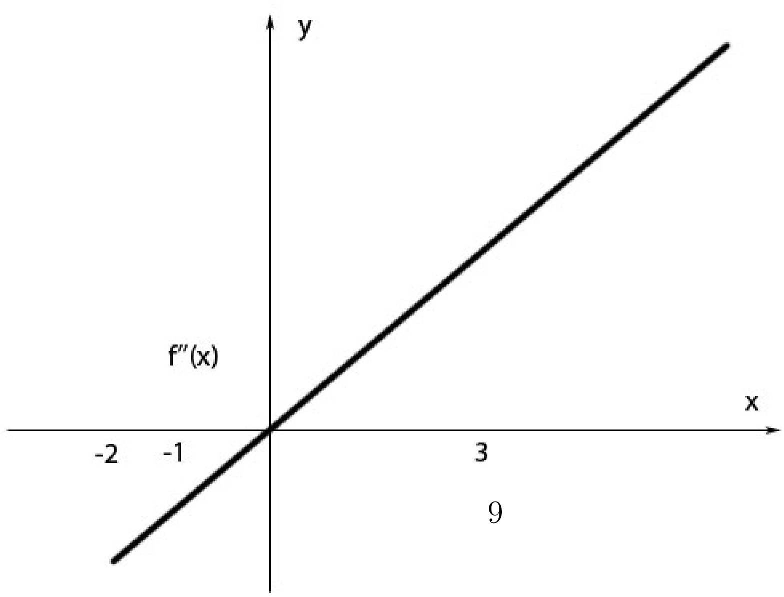
(d) If $p(x)$ is a polynomial such that $p'(x)$ has a simple root at $x = 1$, then p has a relative extremum at $x = 1$.

False- Simple roots do not have tangency to the x axis, so there is no chance for relative extremum here.

7. The of a function $f(x)$ is given. Sketch the graphs of $f'(x)$ and $f''(x)$.




y
f′(x)
x
-2
-1
3


y
f′′(x)
x
-2
-1
3

8. Use the given derivative to find all critical points of f, and at each critical point determine whether a relative max, min or neither occurs. Assume in each case that f is continuous everywhere.

(a) $f'(x) = x^2(x^3 - 5)$

The zeros of $f'(x)$ occur at $x = 0$, and $x = \sqrt[3]{5} \approx 1.25$. These are the critical points. To test for min or max, we can look at the sign of $f'(x)$ by testing $x = -1$, $x = 1$ and $x = 2$.

$$\begin{aligned} f'(-1) &= (-1)^2((-1)^3 - 5) \\ &= 1(-1 - 5) = -6 < 0 \end{aligned}$$

$$\begin{aligned} f'(1) &= 1^2(1^3 - 5) \\ &= 1 - 5 = -4 < 0 \end{aligned}$$

$$\begin{aligned} f'(2) &= (2)^2(2^3 - 5) \\ &= 4(8 - 5) = 12 > 0 \end{aligned}$$

So by the first derivative test, we have a $\boxed{Relative\ Min\ at\ x = \sqrt[3]{5}}$ and $\boxed{Neither\ Relative\ Min\ or\ Max\ at\ x = 0}$

(b) $f'(x) = \frac{2-3x}{\sqrt[3]{x+2}}$

The zeros of $f'(x)$ occur when $2 - 3x = 0$, so at $x = \frac{2}{3}$ there is a critical point. Again, we can test points $x = 0$ and $x = 1$ to determine the sign.

$$f'(0) = \frac{2 - 3(0)}{\sqrt[3]{0 + 2}} = \frac{2}{\sqrt[3]{2}} > 0$$

$$f'(1) = \frac{2 - 3(1)}{\sqrt[3]{1 + 2}} = -\frac{1}{\sqrt[3]{1 + 2}} < 0$$

So by the first derivative test, we have a $\boxed{Relative\ Max\ at\ x = \frac{2}{3}}$

9. Find the relative extrema using the first and second derivative tests.

(a) $$f(x) = 1 + 8x - 3x^2$$

$$f'(x) = 8 - 6x = 0$$
$$8 = 6x$$
$$\frac{8}{6} = \boxed{\frac{4}{3} = x}$$

$$f''(x) = -6 < 0$$

It is clear that $f''(x) < 0$ for every x, so $f(x)$ is always concave down. Thus, there is a $\boxed{\textit{Relative Min at } x = 4/3}$ by the second derivative test.

Now, to use the first derivative test, we need to check the points $x = 1$ and $x = 2$ for the sign of $f'(x)$:

$$f'(1) = 8 - 6(1) = 2 > 0$$
$$f'(2) = 8 - 6(2) = 8 - 12 = -4 < 0$$

So there is a $\boxed{\textit{Relative Max at } x = 4/3}$. By plugging in this x into the original $f(x)$ we obtain the point:

$$\begin{aligned} f(4/3) &= 1 + 8(4/3) - 3(4/3)^2 \\ &= 1 + 32/3 - 3(16/9) \\ &= 3/3 + 32/3 - 16/3 \\ &= 19/3 \end{aligned}$$

So the relative max occurs at the point $\left(\frac{4}{3}, \frac{19}{3}\right)$.

(b) $$f(x) = \sin 2x, \qquad 0 < x < \pi$$

$$\begin{aligned} f'(x) &= 2\cos 2x = 0 \\ \cos 2x &= 0 \\ 2x &= \cos^{-1}(0) \\ 2x &= \pi/2 \\ x &= \pi/4 \end{aligned}$$

$$f''(x) = -4\sin 2x$$

For the first derivative test, we can use the test points $x = \pi/6$, and $x = \pi/3$.

$$\begin{aligned} f'(\pi/6) &= 2\cos(2(\pi/6)) \\ &= 2\cos(\pi/3) \\ &= 2(1/2) = 1 > 0 \end{aligned}$$

$$\begin{aligned} f'(\pi/3) &= 2\cos(2(\pi/3)) \\ &= 2(-1/2) = -1 < 0 \end{aligned}$$

Thus there is a $\boxed{\textit{Relative Max at } x = \pi/4}$

For the second derivative test:

$$\begin{aligned} f''(\pi/4) &= -4\sin(2(\pi/4)) \\ &= -4\sin(\pi/2) \\ &= -4(1) = -4 < 0 \end{aligned}$$

So concave down implies that there is a $\boxed{\textit{Relative Max at } x = \pi/4}$

10. Use any method to find the relative extrema of the function f.

(a)
$$f(x) = x^4 - 4x^3 + 4x^2$$

$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 + 8x \\ &= 4x(x^2 - 3x + 2) \\ &= 4x(x-1)(x-2) \end{aligned}$$

$$f''(x) = 12x^2 - 24x + 8$$

The zeros of $f'(x)$ are $x = 0,\ 1,\ 2$. Use the second derivative test:

$$f''(0) = 8 > 0 \text{ Concave up} \rightarrow \boxed{Rel.\ Min\ x = 0}$$

$$f''(1) = 12 - 24 + 8 = -4 < 0 \text{ Concave Down} \rightarrow \boxed{Rel\ Max\ x = 1}$$

$$f''(2) = 12(2)^2 - 24(2) + 8 = 8 > 0 \text{ Concave up} \rightarrow \boxed{Rel\ Min\ x = 2}$$

(b)
$$\begin{aligned} f(x) &= x^3(x+1)^2 \\ &= x^3(x^2 + 2x + 1) \\ &= x^5 + 2x^4 + x^3 \end{aligned}$$

$$\begin{aligned} f'(x) &= 5x^4 + 8x^3 + 3x^2 \\ &= x^2(5x^2 + 8x + 3) \\ &= x^2(5x+3)(x+1) \end{aligned}$$

$$f''(x) = 20x^3 + 24x^2 + 6x$$

The zeros of $f'(x)$ are $x = 0,\ -3/5,\ -1$. Use the second derivative test:

$$f''(0) = 0 \ \rightarrow \underline{\text{Inconclusive}}, \text{ will use first derivative test later.}$$

$$\begin{aligned} f''(-3/5) &= 20(-3/5)^3 + 24(-3/5)^2 + 6(-3/5) \\ &= -20(27/125) + 24(9/25) - 18/5 \\ &= \frac{-20(27) + 24(9)(5) - 18(25)}{125} \\ &= \frac{-540 + 1080 - 450}{125} \\ &= \frac{90}{125} > 0 \ \text{ Concave Up} \rightarrow \boxed{Rel.\ Min\ x = -3/5} \end{aligned}$$

$$\begin{aligned} f''(-1) &= 20(-1)^3 + 24 - 6 \\ &= -26 + 24 \\ &= -2 < 0 \ \text{ Concave Down} \rightarrow \boxed{Rel.\ Max\ x = -1} \end{aligned}$$

Use the first derivative test for $x = 0$, by testing $f'(-1/2)$ and $f'(1)$:

$$\begin{aligned} f'(-1/2) &= \left(-\frac{1}{2}\right)^2 \left(5\left(-\frac{1}{2}\right) + 3\right)\left(-\frac{1}{2} + 1\right) \\ &= \frac{1}{4}\left(\left(-\frac{5}{2}\right) + 3\right)\left(\frac{1}{2}\right) = \frac{1}{8}\left(\frac{4}{3}\right) > 0 \end{aligned}$$

$$f'(1) = 1(5+3)(1+1) > 0$$

So $f(x)$ is always increasing through $x = 0$ and there is no relative extremum here.

(c) $f(x) = |3x - x^2|$

First determine how to break this function up into a piecewise function. We can factor $f(x)$ as $|x(3-x)|$. So the zeros of this function are at $x = 0$ and $x = 3$.

For $x < 0$: $(-)(+) = -$

For $0 < x < 3$: $(+)(+) = +$

For $x > 3$: $(+)(-) = -$

Now we can rewrite $f(x)$ and find its derivatives:

$$f(x) = \begin{cases} 3x - x^2, & 0 < x < 3 \\ -(3x - x^2), & x < 0,\ x > 3 \end{cases}$$

$$f'(x) = \begin{cases} 3 - 2x, & 0 < x < 3 \\ -3 + 2x, & x < 0,\ x > 3 \end{cases}$$

$$f''(x) = \begin{cases} -2, & 0 < x < 3 \\ 2, & x < 0,\ x > 3 \end{cases}$$

Now $f'(x) = 0$ for $x = \frac{3}{2}$. Since $0 < \frac{3}{2} < 3$ we know that $f''(3/2) = -2 < 0$ and so concave down at this point. Thus there is a $\boxed{\textit{Relative Maximum at } x = \dfrac{3}{2}}$.

We can infer the existence of other extrema by the nature of the absolute value function. We should know that when the absolute value function changes from positive to negative, we get a cusp at that point (*Think* $|x|$ *at 0*). Since this occurs at the zeros of our function, they must be the *lowest* values the function takes. So we have $\boxed{\textit{Relative Minimums at } x = 0 \textit{ and } x = 3}$. However these are only critical points, not stationary, as the derivative does not exist here.