Analysis of Functions: Rational Functions, Cusps, and Vertical Tangents
Solutions To Selected Problems
Calculus 9^{th} Edition Anton, Bivens, Davis

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October 18, 2011

1. Give a graph of the rational function and label the coordinates of the stationary points and inflection points. Show the horizontal and vertical asymptotes and label them their equations. Label points, if any, where the graph crosses a horizontal asymptote.

   (a) $f(x) = \frac{x}{x^2-4}$

      i. Symmetry:

$$\begin{aligned} f(-x) &= \frac{-x}{(-x)^2-4} \\ &= -\frac{x}{x^2-4} \\ &= -f(x) \qquad \rightarrow \text{Origin Symmetry} \end{aligned}$$

      ii. $x,\ y$ Intercepts:

$$\begin{aligned} f(0) &= \frac{0}{0^2-4} = 0 \\ f(x) &= 0 \\ \frac{x}{x^2-4} &= 0 \\ x &= 0 \end{aligned}$$

      iii. Vertical asymptotes: Occur for $f(x)$ when denominator is zero.

$$\begin{aligned} x^2-4 &= 0 \\ (x+2)(x-2) &= 0 \\ x &= 2, \quad x = -2 \end{aligned}$$

      iv. Sign of $f(x)$: Check points $x = -3, -1, 1, 3$

$$\begin{aligned} f(-3) &= \frac{-3}{9-4} = \frac{-3}{5} < 0 \\ f(-1) &= \frac{-1}{1-4} = \frac{-1}{-3} = \frac{1}{3} > 0 \\ f(1) &= \frac{1}{1-4} = -\frac{1}{3} < 0 \\ f(3) &= \frac{3}{9-4} = \frac{3}{5} > 0 \end{aligned}$$

v. End Behavior:

$$\lim_{x\to\infty} \frac{x}{x^2-4} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1-\frac{4}{x^2}} = \frac{0}{1-0} = 0$$

$$\lim_{x\to-\infty} \frac{x}{x^2-4} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{1-\frac{4}{x^2}} = \frac{0}{1-0} = 0$$

vi. Derivatives:

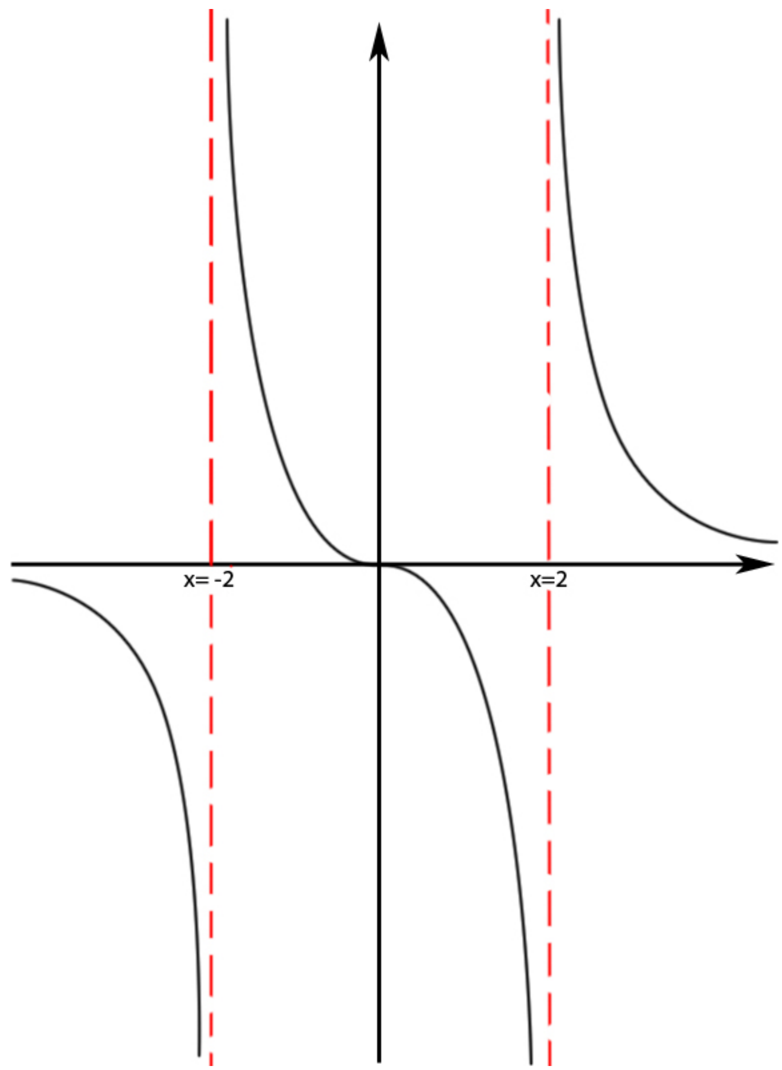
$$f(x) = \frac{x}{x^2-4}$$

$$\begin{aligned} f'(x) &= \frac{(x^2-4)-x(2x)}{(x^2-4)^2} \\ &= \frac{x^2-4-2x^2}{(x^2-4)^2} \\ &= \frac{-(x^2+4)}{(x^2-4)^2} < 0 \qquad \text{Always Decreasing and} \neq 0 \end{aligned}$$

$$\begin{aligned} f''(x) &= -\frac{(x^2-4)^2(2x)-(x^2+4)2(x^2-4)(2x)}{(x^2-4)^4} \\ &= -\frac{(x^2-4)[(x^2-4)(4x)-4x(x^2+2)]}{(x^2-4)^4} \\ &= -\frac{4x^3-16x-4x^3-8x}{(x^2-4)^3} \\ &= -\frac{-24x}{(x^2-4)^3} \\ &= \frac{24x}{(x^2-4)^3} = 0 \quad \textit{when } x = 0 \end{aligned}$$

Interval	Sign of $f'(x)$	Sign of $f''(x)$
$x < -2$	$-$	$-$
$-2 < x < 0$	$-$	$+$
$0 < x < 2$	$-$	$-$
$x > 2$	$-$	$+$

vii. Conclusions and Graph:



(b) $f(x) = \frac{x^2}{x^2+4}$

i. <u>Symmetry:</u>

$$f(-x) = \frac{(-x)^2}{(-x)^2+4}$$
$$= \frac{x^2}{x^2+4} = f(x) \quad \rightarrow \text{ Symmetric w.r.t. } x\text{-axis}$$

ii. <u>$x,\ y$ Intercepts:</u>

$$f(0) = \frac{0^2}{0^2+4} = 0$$

$$f(x) = 0$$
$$\frac{x^2}{x^+4} = 0$$
$$x^2 = 0$$
$$x = 0$$

iii. <u>Vertical asymptotes:</u> By inspection we see that $f(x)$ is continuous for all values of x, so there are no vertical asymptotes on $(-\infty, +\infty)$

iv. <u>Sign of $f(x)$:</u> $f(x) > 0$ on $(-\infty, +\infty)$
Also note that since $x^2 < x^2 + 4$, then $\frac{x^2}{x^2+4} < 1$, so we know that $0 < \frac{x^2}{x^2+4} < 1$.

v. <u>End Behavior:</u>

$$\lim_{x\to\infty} \frac{x^2}{x^2+4} = \lim_{x\to\infty} \frac{1}{1+\frac{4}{x^2}} = \frac{1}{1+0} = 1$$

$$\lim_{x\to-\infty} \frac{x^2}{x^2+4} = \lim_{x\to-\infty} \frac{1}{1+\frac{4}{x^2}} = \frac{1}{1+0} = 1$$

vi. Derivatives:

$$f(x) = \frac{x^2}{x^2+4}$$

$$\begin{aligned} f'(x) &= \frac{(x^2+4)2x - x^2(2x)}{(x^2+4)^2} \\ &= \frac{2x^3+8x-2x^3}{(x^2+4)^2} \\ &= \frac{8x}{(x^2+4)^2} \end{aligned}$$

$$f'(x) = 0 \ when \ x = 0$$

$$\begin{aligned} f''(x) &= 8\frac{(x^2+4)^2(1) - x(2(x^2+4)2x)}{(x^2+4)^4} \\ &= 8\frac{(x^2+4)(x^2+4-4x^2)}{(x^2+4)^4} \\ &= 8\frac{4-3x^2}{(x^2+4)^3} \\ &= -8\frac{3x^2-4}{(x^2+4)^3} \end{aligned}$$

$$\begin{aligned} f''(0) &= -8\frac{3(0)^2-4}{(0+4)^3} \\ &= -8\frac{-4}{4^3} = 8\frac{1}{4^2} > 0 \rightarrow \text{Concave up at } x = 0 \text{ by } 2^{nd} \text{ derivative test} \end{aligned}$$

$$\begin{aligned} f''(x) &= 0 \ when \ 3x^2 - 4 = 0 \\ 3x^2 &= 4 \\ x^2 &= \frac{4}{3} \\ x &= \pm\frac{2}{\sqrt{3}} \end{aligned}$$

$$f\left(\frac{2}{\sqrt{3}}\right) = \frac{\left(\frac{2}{\sqrt{3}}\right)^2}{\left(\frac{2}{\sqrt{3}}\right)^2 + 4}$$
$$= \frac{\frac{4}{3}}{\frac{4}{3}+4} = \frac{\frac{4}{3}}{\frac{16}{3}} = \frac{4}{16} = \frac{1}{4}$$

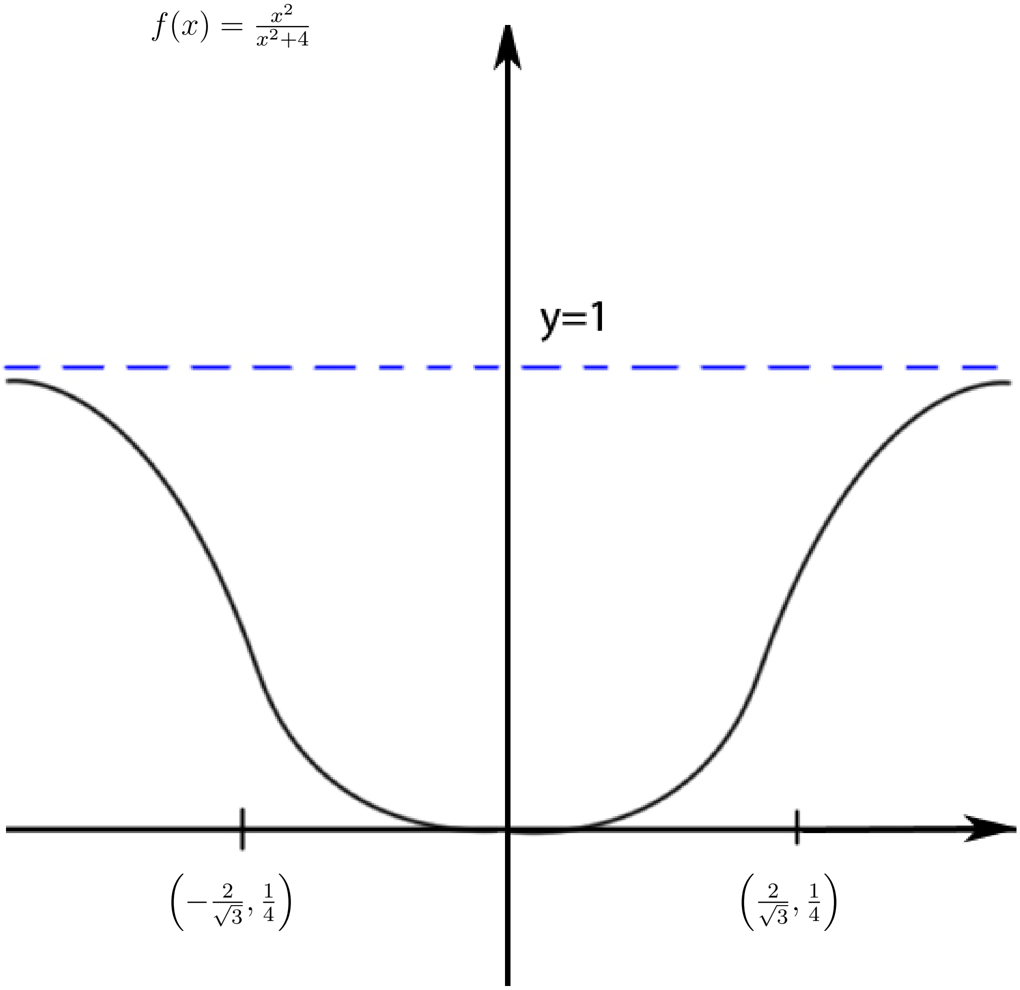
By symmetry, we have $f\left(-\frac{2}{\sqrt{3}}\right) = f\left(\frac{2}{\sqrt{3}}\right) = \frac{1}{4}$.

vii. Conclusions and Graph: Here is a sign chart for our function.

Interval	Sign of $f(x)$	Sign of $f'(x)$	Sign of $f''(x)$
$x < -2/\sqrt{3}$	$+$	$-$	$-$
$-2/\sqrt{3} < x < 0$	$+$	$-$	$+$
$0 < x < 2/\sqrt{3}$	$+$	$+$	$+$
$x > 2/\sqrt{3}$	$+$	$+$	$-$

Inflection points occur at $\left(-\frac{2}{\sqrt{3}}, \frac{1}{4}\right)$ and $\left(\frac{2}{\sqrt{3}}, \frac{1}{4}\right)$. $f(x)$ is always positive and tends towards $y = 1$ as $x \to \pm\infty$.

$f(x) = \frac{x^2}{x^2+4}$



2. Show that $y = x + 3$ is an oblique asymptote of the graph of $f(x) = \frac{x^2}{x-3}$. Sketch the graph of $y = f(x)$ showing the asymptotic behavior.

(a) First we long divide:

$$\begin{array}{r|l} & x+3 \\ \hline x-3 & x^2 \\ & -x^2+3x \\ \hline & 3x \\ & -3x+9 \\ \hline & 9 \end{array}$$

(b)

$$\begin{aligned} f(x) &= x + 3 + \frac{9}{x^2} \\ f(x) - (x+3) &= \frac{9}{x^2} \\ \lim_{x\to\infty} (f(x) - (x+3)) &= \lim_{x\to\infty} \frac{9}{x^2} = 0 \\ f(x) &\to x + 3 \quad x \to \infty \end{aligned}$$

Thus we have shown that $y = x+3$ is an oblique asymptote to the function $\frac{x^2}{x-3}$. To graph the rest of the function we need to due some more analysis.

i. Symmetry:

$$f(-x) = \frac{(-x)^2}{-x-3} = -\frac{x^2}{x+3}$$

There is no symmetry.

ii. $x,\ y$ Intercepts:

$$\begin{aligned} &f(0) = 0 \\ &f(x) = 0 \ when \ x = 0 \end{aligned}$$

iii. Vertical asymptotes: $x = 3$

iv. Sign of $f(x)$:

$$f(2) = \frac{4}{-1} < 0$$
$$f(4) = \frac{16}{1} > 0$$

v. End Behavior: This was already done above.

vi. Derivatives: Take f' and f'' of original $f(x)$.

$$f(x) = \frac{x^2}{x-3}$$

$$\begin{aligned} f'(x) &= \frac{(x-3)(2x) - x^2(1)}{(x-3)^2} \\ &= \frac{x^2 - 6x}{(x-3)^2} \\ &= \frac{x(x-6)}{(x-3)^2} \end{aligned}$$

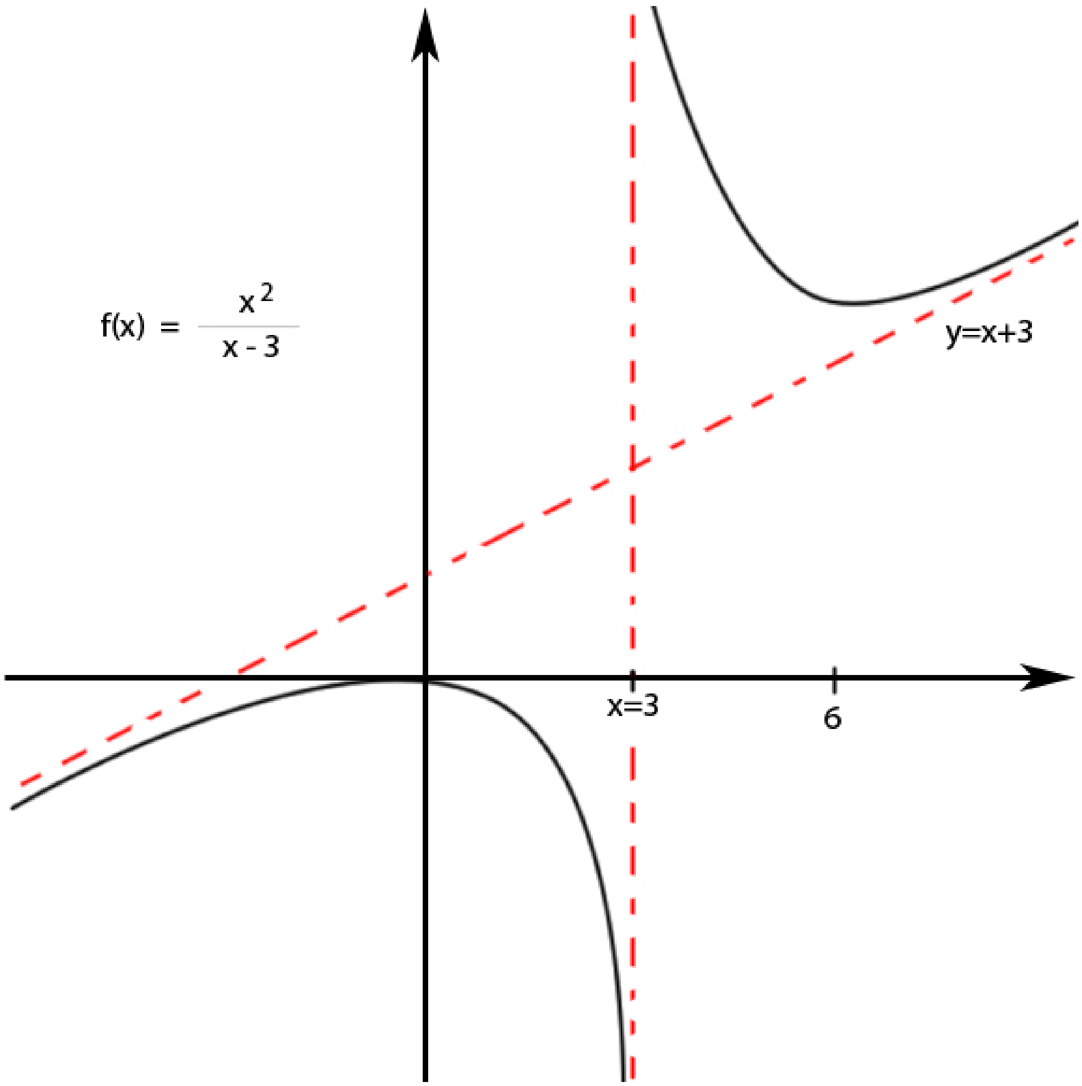
$$f'(x) = 0 \ when \ x = 0, \ x = 6$$

$$\begin{aligned} f''(x) &= \frac{(x-3)^2(2x-6) - (x^2-6x)(2(x-3)(1))}{(x-3)^4} \\ &= \frac{(x-3)[(x-3)(2x-6) - 2x^2 - 12x]}{(x-3)^4} \\ &= \frac{2x^2 - 12x + 18 - 2x^2 - 12x}{(x-3)^3} \\ &= \frac{18}{(x-3)^3} \end{aligned}$$

$$f''(0) = \frac{18}{-3^3} < 0 \quad \text{Concave Down and Rel. Max}$$

$$f''(6) = \frac{18}{3^3} > 0 \quad \text{Concave Up and Rel. Min}$$

vii. Graph:



3. Sketch the graph of the rational function $f(x) = \frac{(x-2)^3}{x^2}$ and label the coordinates of the stationary points and inflection points. Show all possible asymptotes and label them with their equations.

Symmetry: $f(-x) = \frac{(-x-2)^3}{(-x)^2} = \frac{-(x+2)^3}{x^2} \rightarrow$ There is no symmetry.

$x,\ y$ Intercepts: $f(x) = 0$ only when $x = 2$. There is no y intercept as $f(x)$ is undefined at $x = 0$

Vertical asymptotes: $x = 0$

Sign of $f(x)$: It is easy to see that for $x < 2,\ f(x) < 0$ and for $x > 2,\ f(x) > 0$

End Behavior: Rewrite $f(x) = \frac{(x-2)^3}{x^2} = \frac{x^3-6x^2+12x-8}{x^2} = x - 6 + \frac{12}{x} - \frac{8}{x^2}$

Now we can see that $\lim_{x\to\pm\infty} (f(x) - (x-6)) = \lim_{x\to\pm\infty} \frac{12}{x} - \frac{8}{x^2} = 0$

So $f(x) \rightarrow (x-6)\ as\ x \rightarrow \pm\infty$

Thus there is an $\boxed{oblique\ asymptote\ y = x - 6}$

Derivatives:

$$f(x) = \frac{(x-2)^3}{x^2}$$

$$f'(x) = \frac{x^2(3(x-2)^2 - (x-2)^3(2x))}{x^4}$$
$$= \frac{x(x-2)^2(3x-2(x-2))}{x^4}$$
$$= \frac{(x-2)^2(x+4)}{x^3}$$

$$f'(x) = 0 \; when \; x = 2, \; x = -4$$
$$f(2) = 0$$
$$f(-4) = \frac{(-6)^3}{(-4)^2} = \frac{(-2\cdot 3)^3}{(-2^2)^2} = \frac{-2^3 3^3}{2^4} = -\frac{3^3}{2} = -\frac{27}{2}$$

$$f''(x) = \frac{x^3[2(x-2)(x+4)+(x-2)^2(1)] - 3x^2(x-2)^2(x+4)}{x^6}$$

$$= \frac{x^3(x-2)[2(x+4)+(x-2)] - 3x^2(x-2)^2(x+4)}{x^6}$$

$$= \frac{x^2(x-2)[[x(2x+8+x-2)] - 3(x-2)(x+4)]}{x^6}$$

$$= \frac{(x-2)[x(3x+6) - 3(x^2+2x-8)]}{x^4}$$

$$= \frac{(x-2)[3x^2+6x-3x^2-6x+24]}{x^4}$$

$$= \frac{24(x-2)}{x^4}$$

$$f''(x) = 0 \; when \; x = 2$$

Conclusions Note that $f''(4) = \frac{24(-2)}{4^4} < 0$, so by the second derivative test, $f(x)$ is concave down at $x = -4$ and thus there is a $\boxed{\textit{relative max at } x = -4}$

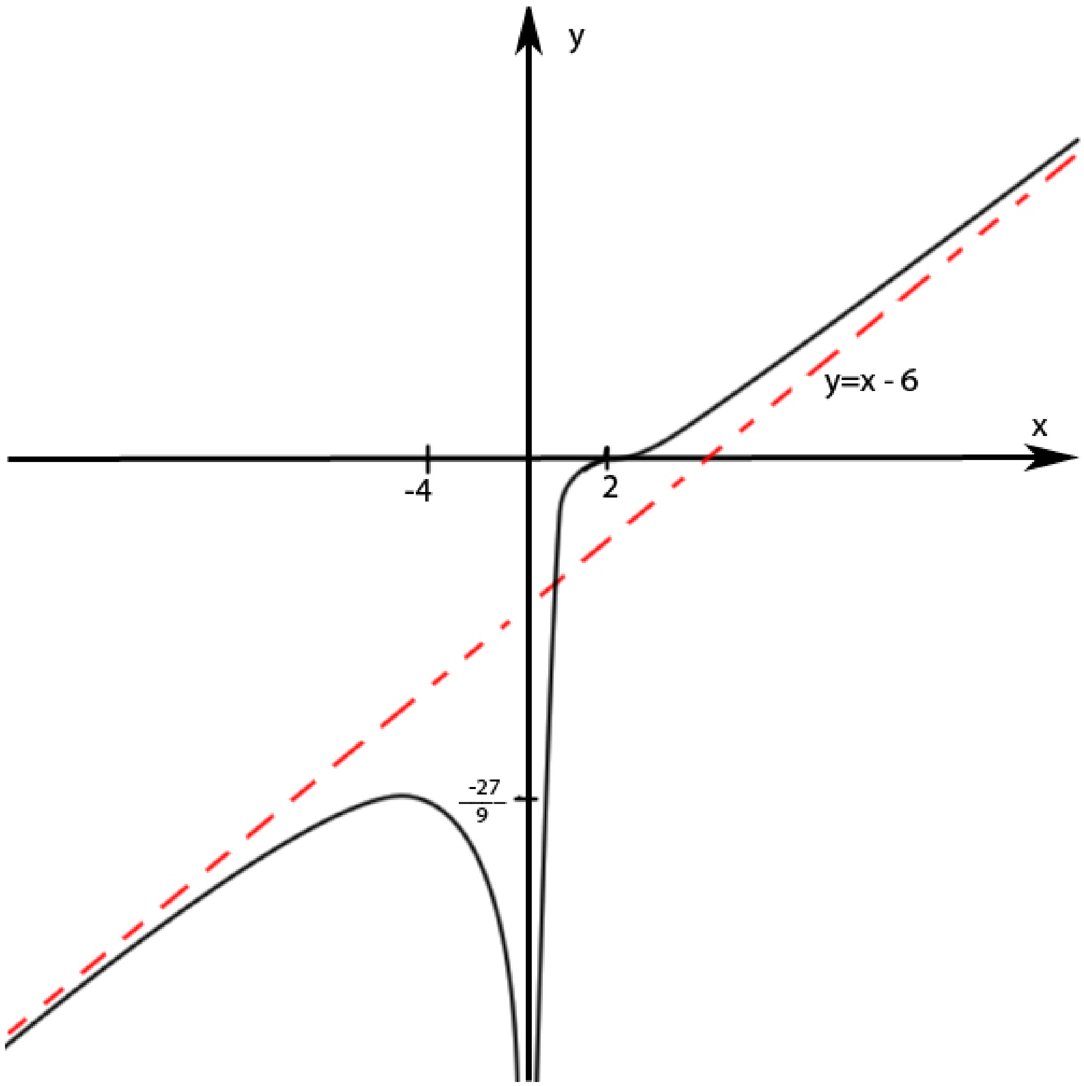
The second derivative test is inconclusive at $x = 2$ as $f''(2) = 0$. Make a sign chart to determine other possible inflections points.

Interval	Sign of $f(x)$	Sign of $f'(x)$	Sign of $f''(x)$
$x < -4$	$-$	$+$	$-$
$-4 < x < 0$	$-$	$-$	$-$
$0 < x < 2$	$-$	$+$	$-$
$x > 2$	$+$	$+$	$+$

We see that there is only one inflection point at $(2, 0)$, which is also a stationary point. The other stationary point occurs at $\left(-4, -\frac{27}{2}\right)$.

Remember that there is an oblique asymptote $y = x - 6$, and a vertical asymptote $x = 0$

Graph of $f(x) = \frac{(x-2)^3}{x^2}$



4. Give a graph of the function $f(x) = 4x^{1/3} - x^{4/3}$ and identify the locations of all relative extrema and inflection points.

$f(x) = 4x^{1/3} - x^{4/3} = x^{1/3}(4 - x)$

We see that $f(x) = 0$ for $x = 0$ and $x = 4$, and that there are no vertical asymptotes. Here is a simple sign chart of $f(x)$ to help us determine where is above or below the x axis:

$$\begin{aligned} x < 0, &\quad f(x) < 0 \\ 0 < x < 4, &\quad f(x) > 0 \\ x > 4, &\quad f(x) < 0 \end{aligned}$$

Now take the first and second derivatives to determine critical points and inflection points.

$$\begin{aligned} f'(x) &= \frac{4}{3}x^{-2/3} - \frac{4}{3}x^{1/3} \\ &= \frac{4}{3}x^{-2/3}(1 - x) \\ &= \frac{4}{3}\frac{1}{\sqrt[3]{x^2}}(1 - x) \end{aligned}$$

$f'(x)$ is undefined at $x = 0$
$f'(x) = 0$ when $x = 1$, with $f(1) = 1^{1/3}(4 - 1) = 3$. Stationary point at $(1, 3)$.

Sign chart for $f'(x)$:

$$\begin{aligned} x < 0 &\quad f'(x) > 0 \\ 0 < x < 1 &\quad f'(x) > 0 \\ x > 1 &\quad f'(x) < 0 \end{aligned}$$

Now the second derivative:

$$f''(x) = -\frac{8}{9}x^{-5/3} - \frac{4}{9}x^{-2/3}$$

$$= -\frac{4}{9}x^{-5/3}(2 + x)$$

$$= -\frac{4}{9}\frac{1}{\sqrt[3]{x^5}}(2 + x)$$

$f''(x)$ is undefined also at $x = 0$
$f''(x) = 0$ when $x = -2$, with $f(-2) = (-2)^{1/3}(4 - (-2)) = -6\sqrt[3]{2}$.
Inflection point at $(-2, -6\sqrt[3]{2})$

Since $f''(1) = -\frac{4}{9}\frac{1}{\sqrt[3]{1^5}}(2 + 1) < 0$, the stationary point is Concave down, and a Relative Max by the second derivative test.

Sign chart for $f''(x)$:

$$x < -6\sqrt[3]{2} \quad f''(x) < 0$$

$$-6\sqrt[3]{2} < x < 0 \quad f''(x) > 0$$

$$x > 0 \quad f''(x) < 0$$

Now it would be good to check the end behavior before sketching the graph.

$$\lim_{x \to \infty} x^{1/3}(4 - x) \to -\infty$$

$$\lim_{x \to -\infty} x^{1/3}(4 - x) \to -\infty$$

Graph of $f(x) = 4x^{1/3} - x^{4/3}$