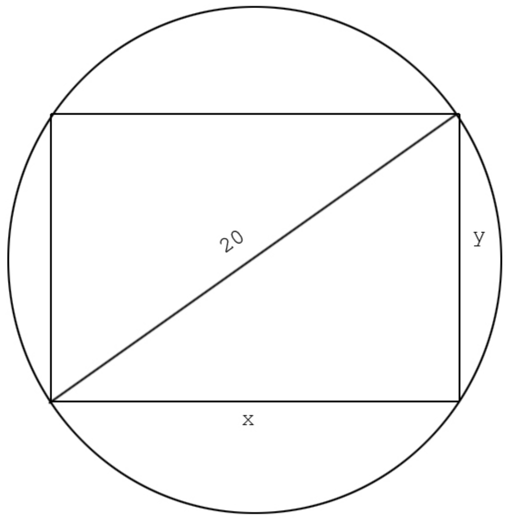
Applied Max and Min
Solutions To Selected Problems
Calculus 9^{th} Edition Anton, Bivens, Davis

Matthew Staley

October 27, 2011

1. Find the dimensions of the rectangle with the maximum area that can be inscribed in a circle of radius 10.



Let x and y be as is shown in the figure above. Since the radius is 10, the hypotenuse of the triangle is the diameter = 20. Then we have the constraint $x^2 + y^2 = 20^2$ and we must maximize area, $A = xy$

$$x^2 + y^2 = 400$$
$$y^2 = 400 - x^2$$
$$y = \sqrt{400 - x^2} \qquad \text{Sub into } A = xy$$

$$A = x\sqrt{400 - x^2} \qquad 0 \leq x \leq 20$$

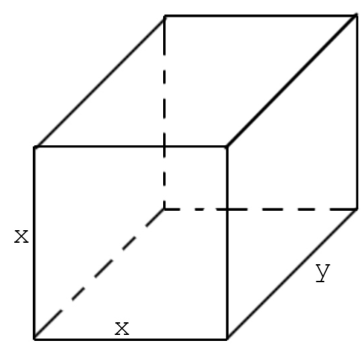
$$\begin{aligned}\frac{dA}{dx} &= \sqrt{400 - x^2} + x\frac{1}{2}\frac{-2x}{\sqrt{400 - x^2}} \\ &= \frac{(400 - x^2) - x^2}{\sqrt{400 - x^2}} \\ &= \frac{2(200 - x^2)}{\sqrt{400 - x^2}} = 0\end{aligned}$$

$$\rightarrow 200 - x^2 = 0$$
$$x^2 = 200$$
$$x = \sqrt{200} = 10\sqrt{2}$$

$$\rightarrow y^2 = 400 - 200$$
$$y^2 = 200$$
$$y = \sqrt{200} = 10\sqrt{2}$$

The dimensions $x = y = 10\sqrt{2}$ show that a square has the maximum area inscribed in a circle.

2. A box with a square base is taller than it is wide. In order to send the box through the U.S. mail, the height of the box and the perimeter of the base can sum to no more than 108 inches. What is the maximum volume for such a box?



Let x and y be as is shown in the figure above. The perimeter of the base is then $4x$ and the height is y. We have the constraint equation $4x + y \leq 108$ and we want to maximize the volume $V = x^2y$.

$$\begin{aligned} \mathit{Let}\ 4x + y &= 108 \\ y &= 108 - 4x \end{aligned}$$

$$\begin{aligned} V = x^2y &= x^2(108 - 4x) \\ &= 108x^2 - 4x^3 \end{aligned}$$

$$\begin{aligned} \frac{dV}{dx} &= (2)180x - (3)4x^2 \\ &= 4x(54 - 3x) = 0 \end{aligned}$$

$$\begin{aligned} \rightarrow 4x &= 0 \\ x &= 0 \end{aligned}$$

$$\begin{aligned} \rightarrow 54 - 3x &= 0 \\ 3x &= 54 \\ x &= 18 \end{aligned}$$

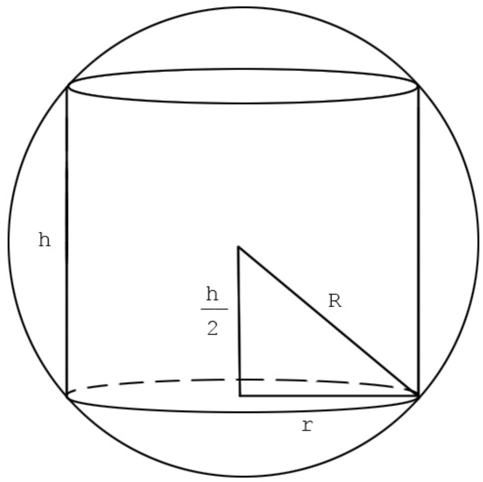
Take $x = 18$ as $x = 0$ is not allowed for our dimension. Substitute x into the equation $y = 108 - 4x$ to find y:

$$\begin{aligned} y &= 108 - 4(18) \\ &= 108 - 72 \\ &= 36 \end{aligned}$$

Therefore our maximum volume will be

$$\begin{aligned} V &= x^2y \\ &= (18)^2(36) \\ &= 324(36) \\ &= 11664 \end{aligned}$$

3. Find the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius R.



Let r and h be the dimensions as shown in the figure above. The volume of the of inscribed cylinder is $V = \pi r^2 h$.

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2$$
$$r^2 = R^2 - \frac{h^2}{4}$$

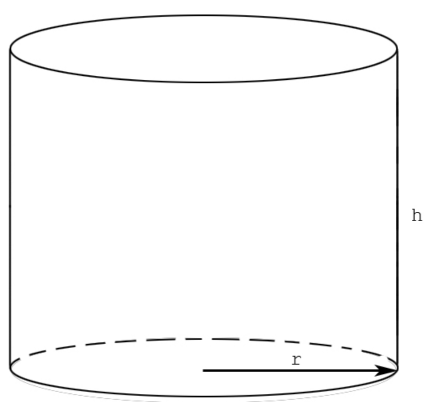
$$\rightarrow V = \pi\left(R^2 - \frac{h^2}{4}\right)h$$
$$= \pi\left(hR^2 - \frac{h^3}{4}\right) \qquad 0 \leq h \leq 2R$$

$$\frac{dV}{dh} = \pi\left(R^2 - \frac{3}{4}h^2\right) = 0$$

$$\rightarrow R^2 - \frac{3}{4}h^2 = 0$$
$$h^2 = \frac{4}{3}R^2$$
$$h = \sqrt{\frac{4}{3}R^2} = \boxed{\frac{2R}{\sqrt{3}}}$$

$$\rightarrow r^2 = R^2 - \frac{h^2}{4}$$
$$= R^2 - \frac{4}{3}\frac{R^2}{4}$$
$$= R^2 - \frac{1}{3}R^2$$
$$= \frac{2}{3}R^2$$
$$r = \boxed{\sqrt{\frac{2}{3}}R}$$

4. A cylindrical can, open on top, is to hold 500 cm^3 of liquid. Find the height and radius that minimizes the amount of material needed to manufacture the can.



The amount of material needed correlates to the surface area of the can. The surface area of the bottom is just the area of the circle πr^2. Since there is an open top, there is no surface area there. The remaining surface area is $2\pi hr$. We have our surface area equation $S = \pi r^2 + 2\pi rh$, which we want to minimize. Our constraint is that $V = \pi r^2 h = 500 \to h = \frac{500}{\pi r^2}$

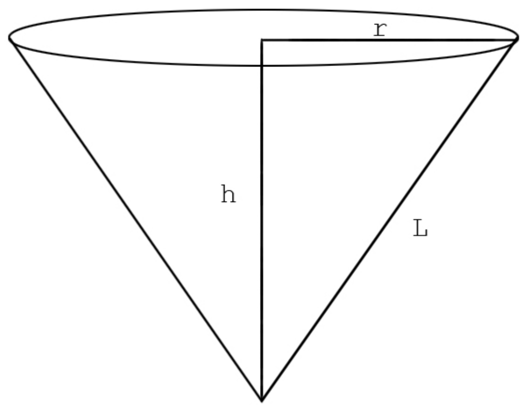
$$\begin{aligned} S &= \pi r^2 + 2\pi r \left(\frac{500}{\pi r^2}\right) \\ &= \pi r^2 + \frac{1000}{r} \qquad r > 0 \end{aligned}$$

$$\frac{dS}{dr} = 2\pi r - \frac{1000}{r^2} = \frac{2\pi r^3 - 1000}{r^2} = 0$$

$$\begin{aligned} \to 2\pi r^3 &= 1000 \\ r^3 &= \frac{500}{\pi} \\ r &= \boxed{\sqrt[3]{\frac{500}{\pi}}} \end{aligned}$$

$$\to h = \frac{500}{\pi r^2} = \frac{500}{\pi}\left(\frac{\pi}{500}\right)^{2/3} = \boxed{\sqrt[3]{\frac{500}{\pi}}}$$

5. A cone shaped paper drinking cup is to hold 100 cm^3 of water. Find the height and radius of the cup that will require the least amount of paper. Let $r,\ h,\ L$



be as shown in the figure above. The area of the paper is then $A = \pi r L = \pi r\sqrt{r^2+h^2}$ since $L^2 = r^2 + h^2$. We also have the constraint $V = \frac{1}{3}\pi r^2 h = 100 \ \rightarrow\ h = \frac{300}{\pi r^2}$. Sub this into A:

$$A = \pi r\sqrt{r^2 + \left(\frac{300}{\pi r^2}\right)^2} = \pi r\sqrt{r^2 + \frac{90000}{\pi^2 r^4}}$$

To simplify computations, let $S = A^2$.

$$\rightarrow S = \pi^2 r^2\left(r^2 + \frac{90000}{\pi^2 r^4}\right) = \pi^2 r^4 + \frac{90000}{r^2}$$

$$\frac{dS}{dr} = 4\pi^2 r^3 - \frac{180000}{r^3} = \frac{4\pi^2 r^6 - 180000}{r^3} = 0$$

$$\rightarrow 4\pi^2 r^6 - 180000 = 0$$
$$4\pi^2 r^6 = 180000$$
$$r^6 = \frac{180000}{4\pi^2} = \frac{45000}{\pi^2}$$

$$\rightarrow r = \boxed{\sqrt[6]{\frac{45000}{\pi^2}}}$$

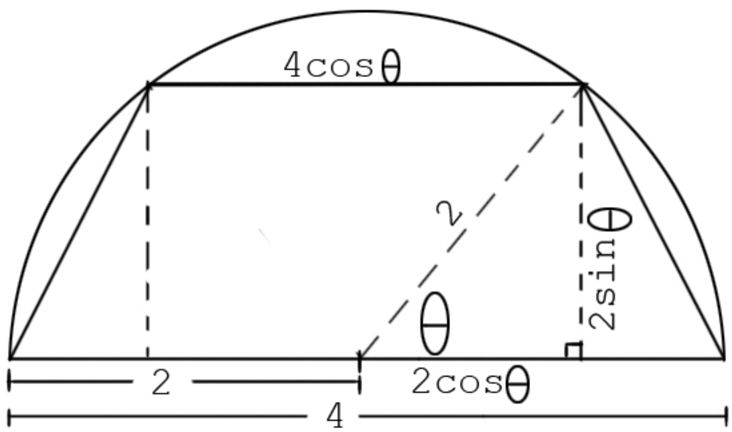
$$\rightarrow h = \frac{300}{\pi r^2}$$

$$= \frac{300}{\pi}\left(\frac{\pi^2}{45000}\right)^{2/6}$$

$$= \frac{300}{\pi}\left(\frac{\pi^2}{45000}\right)^{1/3}$$

$$h = \boxed{\frac{300}{\pi}\sqrt[3]{\frac{\pi^2}{45000}}}$$

6. A trapezoid is inscribed in a semicircle of radius 2 so that one side is along the diameter. Find the maximum possible are for the trapezoid.



By the figure above the area of the trapezoid is :

$$\begin{aligned} A &= \frac{1}{2}(b_1 + b_2)h \\ &= \frac{1}{2}(4 + 4\cos(\theta))(\sin(\theta)) \\ &= 4(\sin(\theta) + \sin(\theta)\cos(\theta)) \qquad 0 \le \theta \le \pi/2 \end{aligned}$$

$$\begin{aligned} \frac{dA}{d\theta} &= 4(\cos(\theta) - \sin^2(\theta) + \cos^2(\theta)) \\ &= 4(\cos(theta) - [1 - \cos^2(\theta)] + \cos^2(\theta)) \\ &= 4(2\cos^2(\theta) + \cos(\theta) - `1) \\ &= 4(2\cos(\theta) - 1)(\cos(\theta) + 1) \\ &= 0 \ when \ \boxed{\theta = \pi/3} \ for \ 0 < \theta < \pi/2 \end{aligned}$$

So the maximum area occurs when $\theta = \pi/3$ which gives :

$$\begin{aligned} A &= 4(\sin(\theta) + \sin(\theta)\cos(\theta)) \\ &= 4(\sin(\pi/3) + \sin(\pi/3)\cos(\pi/3)) \\ &= 4(\sqrt{3}/2 + \sqrt{3}/2(1/2)) \\ &= 2\sqrt{3} + \sqrt{3} \\ &= \boxed{3\sqrt{3}} \end{aligned}$$