Recall: $\frac{d}{dx} [3^x] = 3^x \ln 3$

Using Logarithmic Differentiation

$\frac{d}{dx} [3^x]$

Let $y = 3^x \Rightarrow \ln y = \ln 3^x$

$\ln y = x \ln 3$

$\frac{d}{dx} [\ln y] = \frac{d}{dx} [x \ln 3]$

$\frac{1}{y} \frac{dy}{dx} = \ln 3$

$\frac{dy}{dx} = y \cdot \ln 3$

$\frac{dy}{dx} = 3^x \ln 3$

Notice that $3^x = e^{\ln 3^x}$

$3^x = e^{x \ln 3}$

$\frac{d}{dx} [3^x] = \frac{d}{dx} [e^{x \ln 3}]$

$\frac{d}{dx} [3^x] = e^{x \ln 3} \cdot \ln 3$

$\frac{d}{dx} [3^x] = 3^x \cdot \ln 3$
Ex. 1: Find any horizontal tangents on $f(x) = x^x > 0$

$y = x^x$

$\ln y = \ln x^x$

$\ln y = x \ln x$

$$\frac{dy}{dx} = \frac{d}{dx} [x \ln x]$$

$$\frac{1}{y} \frac{dy}{dx} = \ln x + x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y (\ln x + 1)$$

$$\frac{dy}{dx} = x^x (\ln x + 1)$$

$\therefore f'(x) = x^x (\ln x + 1)$

Setting $f'(x) = 0$

$\ln x + 1 = 0$

$\ln x = -1$

$-1 = x$

$\frac{1}{e} = x$
Example 2: Find any $x$-values that make
\[ \frac{dy}{dx} \] equal to 0, given $y = (\ln x)^{\ln x}$, $x > 1$

\[
\ln y = \ln [(\ln x)^{\ln x}]
\]
\[
\ln y = \ln x \cdot \ln [\ln x]
\]
\[
\frac{d}{dx} [\ln y] = \frac{d}{dx} [\ln x \cdot \ln [\ln x]]
\]
\[
\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} \cdot \ln [\ln x] + \ln x \cdot \frac{1}{\ln x} \cdot \frac{1}{x}
\]
\[
\frac{1}{y} \cdot \frac{dy}{dx} = \frac{\ln (\ln x)}{x} + \frac{1}{x}
\]
\[
\frac{dy}{dx} = y \left[ \frac{\ln (\ln x) + 1}{x} \right]
\]
\[
\frac{dy}{dx} = (\ln x)^{\ln x} \left[ \frac{\ln (\ln x) + 1}{x} \right]
\]

Setting $\frac{dy}{dx} = 0$, we get

\[
\ln (\ln x) + 1 = 0
\]
\[
\ln (\ln x) = -1
\]
\[
e^{-1} = \ln x
\]
\[
e^{-\ln x} = x
\]
\[
e^{\ln x} = x
\]
\[
e = e^{\ln x}
\]
\[
\frac{1}{e} = \ln x
\]
\[
e \cdot \ln x = x
\]
\[
e^{\ln x} = x
\]
\[
e = e\]
Example 3: Find \( \frac{dy}{dx} \)

\[ y = \pi \sin x \]

\[ \ln y = \ln \pi \sin x \]

\[ \frac{d}{dx} \left[ \ln y \right] = \frac{d}{dx} \left[ \ln \pi \sin x \right] = \pi \sin x \cdot \ln \pi \cdot \cos x \]

\[ \frac{d}{dx} [\ln y] = \frac{d}{dx} [\sin x \cdot \ln \pi] \]

\[ \frac{1}{y} \frac{dy}{dx} = \ln \pi \cdot \cos x \]

\[ \frac{dy}{dx} = y \cdot \ln \pi \cdot \cos x \]

\[ \frac{dy}{dx} = \pi \sin x \cdot \ln \pi \cdot \cos x \]

Example 4: Integrate

a) \( \int x^2 e^{-2x^3} \, dx \)

\[ u = -2x^3 \]

\[ du = -6x^2 \, dx \]

\[ \frac{-1}{6} du = x^2 \, dx \]

\[ \frac{-1}{6} \int e^u \, du \]

\[ \frac{-1}{6} \left( e^u + C \right) \]

\[ \frac{-2x^3}{6} + C \]

b) \( \int_{\ln 3}^{e^{\ln 3}} \frac{e^x}{e^x + 4} \, dx \)

\[ u = e^x + 4 \]

\[ du = e^x \, dx \]

\[ x = \ln 3 \rightarrow e^{\ln 3} + 4 = 7 \]

\[ x = -\ln 3 \rightarrow e^{-\ln 3} + 4 = \frac{49}{3} \]

\[ \int_{\ln 3}^{e^{\ln 3}} \frac{1}{u} \, du \]

\[ = \ln u \bigg|_{\ln 3}^{e^{\ln 3}} \]

\[ = \ln 7 - \ln \frac{49}{3} \]

\[ = \ln \frac{21}{\frac{49}{3}} = \ln \left( \frac{21}{13} \right) \]

\[ e^{-\ln 3} \rightarrow e^{\ln 3^{-1}} \rightarrow e^{\frac{4}{3}} \rightarrow \frac{1}{3} \]
Example 5: Solve the following equation for \( x \).

\[
\frac{e^x - e^{-x}}{2} = 1
\]

\[
e^x - e^{-x} = 2
\]

\[
e^x - \frac{1}{e^x} = 2 \quad \text{LCD: } e^x
\]

\[
e^{2x} - 1 = 2e^x
\]

\[
e^{2x} - 2e^x - 1 = 0
\]

\[
(e^x)^2 - 2e^x - 1 = 0
\]

Let \( u = e^x \)

\[
u^2 - 2u - 1 = 0
\]

\[
u^2 - 2u = 1 \quad \text{"Complete the square"}
\]

\[
u^2 - 2u + 1 = 1 + 1
\]

\[
(u - 1)^2 = 2
\]

\[
\sqrt{(u - 1)^2} = \pm \sqrt{2}
\]

\[
u - 1 = \pm \sqrt{2}
\]

\[
u = 1 \pm \sqrt{2}
\]

\[
e^x = 1 \pm \sqrt{2}
\]

\[
x = \ln (1 + \sqrt{2})
\]