Derivatives of Inverse Functions; Derivatives and Integrals Involving Exponential Functions
Solutions To Selected Problems
Calculus 9^{th} Edition Anton, Bivens, Davis

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1. Let $f(x) = x^5 + x^3 + x$.

   (a) Show that f is one-to-one and confirm that $f(1) = 3$.

   $$f(x) = x^5 + x^3 + x$$
   $$f(1) = 1^5 + 1^3 + 1 = 3$$
   $$f'(x) = 5x^4 + 3x^2 + 1 > 0 \quad \textit{For all } x.$$

   By Theorem 6.3.1, $f(x)$ is one-to-one.

   (b) Find $(f^{-1})'(3)$.

   $$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(1)} = \frac{1}{5(1)^4 + 3(1)^2 + 1} = \boxed{\frac{1}{9}}$$

3. Find $(f^{-1})'(x)$ using formula 2

$$f(x) = \frac{2}{x+3}$$
$$f'(x) = -\frac{2}{(x+3)^2}$$

To find the inverse, switch x and y and then solve for y:

$$x = \frac{2}{y+3}$$

$$(y+3)x = 2$$

$$y + 3 = \frac{2}{x}$$

$$y = f^{-1}(x) = \frac{2}{x} - 3$$

Now using formula 2:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'\left(\frac{2}{x} - 3\right)}$$

$$= \frac{1}{\frac{-2}{\left[\left(\frac{2}{x}-3\right)+3\right]^2}} = \frac{1}{\frac{-2}{\left(\frac{2}{x}\right)^2}} = \frac{\left(\frac{2}{x}\right)^2}{-2} = -\frac{4}{2x^2} = \boxed{-\frac{2}{x^2}}$$

7. Find the derivative of f^{-1} by using formula 3.

$$y = 5x^3 + x - 7$$

$$\rightarrow x = 5y^3 + y - 7$$

$$\frac{d}{dy}(x) = \frac{d}{dy}(5y^3 + y - 7)$$

$$\frac{dx}{dy} = \frac{1}{dy/dx} = 15y^2 + 1$$

$$\boxed{\frac{dy}{dx} = (f^{-1})'(y) = \frac{1}{15y^2 + 1}}$$

15. $$y = e^{7x}$$

$$\frac{dy}{dx} = e^{7x} \cdot 7 = \boxed{7e^{7x}}$$

19. $$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{dy}{dx} = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{e^{2x} + e^0 + e^0 + e^{-2x} - (e^{2x} - e^0 - e^0 + e^{-2x})}{(e^x + e^{-x})^2}$$

$$= \frac{e^{2x} + 1 + 1 + e^{-2x} - e^{2x} + 1 + 1 - e^{-2x}}{(e^x + e^{-x})^2}$$

$$= \boxed{\frac{4}{(e^x + e^{-x})^2}}$$

21. $$y = e^{x\tan(x)}$$

$$\frac{dy}{dx} = e^{x\tan(x)} \cdot \frac{d}{dx}(x\tan(x))$$

$$= \boxed{e^{x\tan(x)} \cdot (\tan(x) + x\sec^2(x))}$$

25. $$y = \ln(1 - x^{-x})$$

$$\frac{dy}{dx} = \frac{1}{1 - xe^{-x}} \cdot (-(e^{-x} + xe^{-x}(-1)))$$

$$= \frac{-e^{-x} + xe^{-x}}{1 - xe^{-x}}$$

$$= \frac{-e^{-x}(1-x)}{1 - xe^{-x}} = \frac{-(1-x)}{e^x(1 - xe^{-x})} = \frac{-1+x}{e^x - xe^0} = \boxed{\frac{x-1}{e^x - x}}$$

29. Find $f'(x)$ by formula 7 and then by logarithmic differentiation.

Using formula 7 :

$$f(x) = \pi^{\sin(x)}$$

$$f'(x) = \pi^{\sin(x)} \ln(\pi) \frac{d}{dx}(\sin(x)) = \boxed{\pi^{\sin(x)} \ln(\pi) \cos(x)}$$

Using logarithmic differentiation, let $f(x) = y$:

$$y = \pi^{\sin(x)}$$

$$\ln y = \ln\left(\pi^{\sin(x)}\right) = \sin(x)\ln(\pi)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sin(x)\ln(\pi))$$

$$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(\pi) + \sin(x)\cdot(0)$$

$$\frac{dy}{dx} = y \cdot \ln(\pi)\cos(x) = \boxed{\left[\pi^{\sin(x)}\right] \ln(\pi)\cos(x)}$$

31. Find dy/dx using logarithmic differentiation.

$$y = (x^3 - 2x)^{\ln(x)}$$

$$\ln y = \ln\left((x^3 - 2x)^{\ln(x)}\right)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\ln(x) \cdot \ln(x^3 - 2x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\ln(x^3 - 2x)}{x} + \frac{(3x^2 - 2)\ln(x)}{x^3 - 2x}$$

$$\frac{dy}{dx} = y \cdot \frac{\ln(x^3 - 2x)}{x} + \frac{(3x^2 - 2)\ln(x)}{x^3 - 2x}$$

$$\boxed{\frac{dy}{dx} = (x^3 - 2x)^{\ln(x)}\left(\frac{\ln(x^3 - 2x)}{x} + \frac{(3x^2 - 2)\ln(x)}{x^3 - 2x}\right)}$$

43. Let $f(x) = x^4 + x^3 + 1,\ 0 \le x \le 2.$

(a) Show that f is one-to-one.

$$f'(x) = 4x^3 + 3x^2 = x^2(4x + 3) = 0 \quad for\ x = 0,\ -\frac{3}{4}$$
$$f'(x) > 0\ on\ (0, 2)$$

Thus f is one-to-one.

(b) Let $g(x) = f^{-1}(x)$ and define $F(x) = f(2g(x))$. Find an equation for the tangent line to $y = F(x)$ at $x = 3$

We are asked to find $y - F(3) = F'(3)(x - 3)$. We need to first find $F(3)$ and $F'(3)$

$$F(3) = f(2g(3)) = f(2 \cdot 1) = f(2) = 2^4 + 2^3 + 1 = 16 + 8 + 1 = 25$$

$$F'(x) = f'(2g(x)) \cdot 2g'(x) = 2f'(2g(x))g'(x)$$
$$F'(3) = 2f'(2g(3))g'(3)$$

By inspection, $f(1) = 3$, so $1 = f^{-1}(3) = g(3)$ and

$$g'(3) = (f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(1)} = \frac{1}{7}$$

$$\rightarrow F'(3) = 2f'(2)\left(\frac{1}{7}\right) = 2(44)\frac{1}{7} = \frac{88}{7}$$

The line tangent to $F(x)$ at the point $(3, 25)$ has the equation:

$$\begin{aligned} y - 25 &= \frac{88}{7}(x - 3) \\ y &= \frac{88}{7}x - \frac{264}{7} + 25 \\ &= \frac{88}{7}x - \frac{264 - 175}{7} \\ &= \boxed{\frac{88}{7}x - \frac{89}{7}} \end{aligned}$$