The definition of Area as a Limit; Sigma Notation
Solutions To Selected Problems
Calculus 9^{th} Edition Anton, Bivens, Davis

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1. Evaluate

   (a) $\sum_{k=1}^{3} k^3 = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = \boxed{36}$

   (b) $\sum_{n=0}^{5} 1 = 1 + 1 + 1 + 1 + 1 + 1 = \boxed{6}$

   (c) $\sum_{k=0}^{4} (-2)^k = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 1 - 2 + 4 - 8 + 16 = \boxed{11}$

   (d) $\sum_{k=0}^{10} \cos(k\pi)$

$$= \cos(0) + \cos(\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) + \cos(5\pi) + \cos(6\pi) + \cos(7\pi) + \cos(8\pi) + \cos(9\pi) + \cos(10\pi)$$

$$= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1$$
$$= \boxed{1}$$

2. Write each expression in sigma notation but not evaluate.

   (a) $1 + 2 + 3 + \cdots + 10 = \boxed{\sum_{n=1}^{10} n}$

   (b) $2 + 4 + 6 + 8 + \cdots + 100 = \boxed{\sum_{k=1}^{50} 2k}$

   (c) $1 + 3 + 5 + 7 + 9 + 11 + \cdots + 99 = \boxed{\sum_{k=1}^{50} 2k - 1}$

3. Use Theorem 4.4.2 to evaluate the sums.

(a) $\sum_{k=1}^{100} k = \frac{100(101)}{2} = 50(101) = \boxed{5050}$

(b) $\sum_{k=1}^{30} k(k-2)(k+2)$

$$
\begin{aligned}
&= \sum_{k=1}^{30} k(k^2-4) \\
&= \sum_{k=1}^{30} k^3 - 4k \\
&= \sum_{k=1}^{30} k^3 - 4\sum_{k=1}^{30} k \\
&= \left(\frac{30(31)}{2}\right)^2 - 4\left(\frac{30(31)}{2}\right) \\
&= \frac{30(31)}{2}\left(\frac{30(31)}{2} - 4\right) \\
&= \frac{930}{2}\left(\frac{930}{2} - 4\right) \\
&= 465(465-4) = 465(461) = \boxed{214365}
\end{aligned}
$$

4. Express the sums in closed form.

(a)
$$\sum_{k=1}^{n} \frac{3k}{n} = \frac{3}{n}\sum_{k=1}^{n} k = \frac{3}{n}\left(\frac{n(n+1)}{2}\right) = \boxed{\frac{3}{2}(n+1)}$$

(b)
$$\sum_{n=1}^{n-1} \frac{k^2}{n} = \frac{1}{n} \cdot \frac{(n-1)((n-1)-1)(2(n-1)+1)}{6}$$
$$= \frac{(n-1)(n)(2n-1)}{6n} = \boxed{\frac{(n-1)(2n-1)}{6}}$$

(c)
$$\sum_{k=1}^{n-1} \frac{k^3}{n^2} = \frac{1}{n^2}\left(\frac{(n-1)n}{2}\right)^2 = \frac{(n-1)^2 n^2}{4n^2} = \boxed{\frac{(n-1)^2}{4}}$$

5. Given $f(x) = 3x + 1; \quad [2, 6]$

Divide the interval into $n = 4$ subintervals of equal length and then compute

$$\sum_{k=1}^{4} f(x_k^*) \triangle x$$

with x_k^* as (a) the left endpoint of each subinterval, and (b) the right endpoint of each subinterval.

(a) Left endpoint:

Each subinterval has length $\triangle x = \frac{6-2}{4} = \frac{4}{4} = 1$
Left endpoints will be $x_k^* = 2 + (k-1)1 = 1 + k$

$$\begin{aligned}\sum_{k=1}^{4} f(x_k^*) \triangle x &= \sum_{k=1}^{4} f(1+k)(1) \\ &= f(1+1) + f(1+2) + f(1+3) + f(1+4) \\ &= f(2) + f(3) + f(4) + f(5) \\ &= (3(2)+1) + (3(3)+1) + (3(4)+1) + (3(5)+1) \\ &= 7 + 10 + 13 + 16 = \boxed{46}\end{aligned}$$

(b) Right endpoint:

Each subinterval is still $\triangle x = 1$
Right endpoints will be $x_k^* = 2 + k(1) = 2 + k$

$$\begin{aligned}\sum_{k=1}^{4} f(x_k^*) \triangle x &= \sum_{k=1}^{4} f(2+k) \\ &= f(1+2) + f(1+3) + f(1+4) + f(1+5) \\ &= f(3) + f(4) + f(5) + f(6) \\ &= (3(3)+1) + (3(4)+1) + (3(5)+1) + (3(6)+1) \\ &= 10 + 13 + 16 + 19 = \boxed{58}\end{aligned}$$

6. Use definition 4.4.3 with x_k^* as the *right* endpoint of each subinterval to find the area under the curve $y = x/2$ over the interval $[1, 4]$.

$$f(x) = \frac{x}{2}; \quad [1,4]$$

$$\triangle x = \frac{4-1}{n} = \frac{3}{n}$$

$$x_k^* = 1 + k \triangle x = 1 + k\left(\frac{3}{n}\right)$$

$$\sum_{k=1}^{n} f(x_k^*) \triangle x = \sum_{k=1}^{n} \left(\frac{x_k^*}{2}\right)\left(\frac{3}{n}\right) \qquad (\textit{Write in closed form, then take the limit})$$

$$= \frac{1}{2}\sum_{k=1}^{n}\left(1 + \frac{3}{n}\right)\left(\frac{3}{n}\right)$$

$$= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{3}{n} + \frac{9k}{n^2}\right)$$

$$= \frac{3}{2}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) + \frac{9}{2}\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right)$$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*) \triangle x$$

$$= \lim_{n\to\infty}\left(\frac{3}{2}\right)\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) + \lim_{n\to\infty}\left(\frac{9}{2}\right)\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right)$$

$$= \frac{3}{2}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} 1\right) + \frac{9}{2}\left(\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} k\right)$$

$$= \frac{3}{2}(1) + \frac{9}{2}\left(\frac{1}{2}\right)$$

$$= \frac{6}{4} + \frac{9}{4} = \boxed{\frac{15}{4}}$$

7. Use definition 4.4.3 with x_k^* as the *left* endpoint of each subinterval to find the area under the curve $y = 9 - x^2$ over the interval $[0, 3]$.

$$f(x) = 9 - x^2; \quad [0,3]$$
$$\triangle x = \frac{3-0}{n} = \frac{3}{n}$$
$$x_k^* = 0 + (k-1) \triangle x = \frac{3}{n}(k-1)$$

$$\begin{aligned}
\sum_{k=1}^{n} f(x_k^*) \triangle x &= \sum_{k=1}^{n} (9 - (x_k^*)^2) \frac{3}{n} \\
&= \frac{3}{n} \sum_{k=1}^{n} \left(9 - \left(\frac{3}{9}\right)^2 (k-1)^2 \right) \\
&= \frac{3}{n} \sum_{k=1}^{n} 9 - \left(\frac{3}{n}\right)^3 \sum_{k=1}^{n} (k-1)^2 \\
&= \frac{3(9)}{n} \sum_{k=1}^{n} 1 - \left(\frac{3}{n}\right)^3 \sum_{k=1}^{n} (k^2 - 2k + 1) \\
&= \frac{27}{n} \sum_{k=1}^{n} 1 - \left(\frac{3}{n}\right)^3 \left[\sum_{k=1}^{n} k^2 - 2 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \right] \\
&= 27 \left(\frac{1}{n} \sum_{k=1}^{n} 1 \right) - 3^3 \left(\frac{1}{n^3} \sum_{k=1}^{n} k^2 \right) - \frac{27(2)}{n} \left(\frac{1}{n^2} \sum_{k=1}^{n} k \right) + \frac{27}{n^2} \left(\frac{1}{n} \sum_{k=1}^{n} 1 \right)
\end{aligned}$$

$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \triangle x$

$$= 27 \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} 1 \right) - 27 \lim_{n\to\infty} \left(\frac{1}{n^3} \sum_{k=1}^{n} k^2 \right) - \lim_{n\to\infty} \left(\frac{27(2)}{n} \right) \left(\frac{1}{n^2} \sum_{k=1}^{n} k \right) + \lim_{n\to\infty} \left(\frac{27}{n^2} \right) \left(\frac{1}{n} \sum_{k=1}^{n} 1 \right)$$

$$= 27 - \left(27\tfrac{1}{3} - (0)\tfrac{1}{2} + (0)(1) \right) = 27 - 9 = \boxed{18}$$

Note that limit of the product is the product of the limits ONLY *if each term has a limit that exists and is finite. (See Theorem 1.2.2, page 63)*

8. Use definition 4.4.3 with x_k^* as the *midpoint* of each subinterval to find the area under the curve $y = x^2$ over the interval $[0, 1]$.

$$f(x) = x^2 \quad [0,1]$$

$$\triangle x = \frac{1}{n}$$

$$x_k^* = 0 + \left(k - \frac{1}{2}\right)\frac{1}{n} = \frac{1}{n}\left(k - \frac{1}{2}\right)$$

$$\begin{aligned}
\sum_{k=1}^{n} f(x_k^*) \triangle x &= \sum_{k=1}^{n} (x_k^*)^2 \frac{1}{n} \\
&= \sum_{k=1}^{n} \left(\frac{1}{n}\left(k - \frac{1}{2}\right)\right)^2 \frac{1}{n} \\
&= \sum_{k=1}^{n} \frac{1}{n^2}\left(k - \frac{1}{2}\right)^2 \frac{1}{n} \\
&= \frac{1}{n^3} \sum_{k=1}^{n} \left(k^2 - k + \frac{1}{4}\right) \\
&= \frac{1}{n^3} \sum_{k=1}^{n} k^2 + \frac{1}{n}\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) + \frac{1}{4n^2}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)
\end{aligned}$$

$$\begin{aligned}
\rightarrow A &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*) \triangle x \\
&= \lim_{n\to\infty} \left(\frac{1}{n^3}\sum_{k=1}^{n} k^2\right) + \lim_{n\to\infty}\left(\frac{1}{n}\right)\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) + \lim_{n\to\infty}\left(\frac{1}{4n^2}\right)\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) \\
&= \frac{1}{3} - 0\left(\frac{1}{2}\right) + 0(1) = \boxed{\frac{1}{3}}
\end{aligned}$$