The Fundamental Theorem of Calculus
Solutions To Selected Problems
Calculus 9^{th} Edition Anton, Bivens, Davis

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November 7, 2011

1. Find the area under the curve $y = f(x)$ over the stated interval.

   (a) $f(x) = x^3; \quad [2, 3]$

$$\begin{aligned}\int_2^3 x^3\,dx &= \frac{1}{4}x^4\Big|_2^3\\ &= \frac{1}{4}(3^4 - 2^4)\\ &= \frac{1}{4}(81 - 16) = \boxed{\frac{65}{4}}\end{aligned}$$

   (b) $f(x) = 3\sqrt{x}; \quad [1, 4]$

$$\begin{aligned}\int_1^4 3\sqrt{x}\,dx &= 3\int_1^4 x^{1/2}\,dx\\ &= 3\frac{2}{3}x^{3/2}\Big|_1^4\\ &= 2(4^{3/2} - 1^{3/2})\\ &= 2(8 - 1) = 2(7) = \boxed{14}\end{aligned}$$

2. Find all the values x^* in the stated interval that satisfy Equation (8) in the Mean-Value Theorem for Integrals (4.6.2).

   (a) $f(x) = \sqrt{x}; \quad [0, 3]$

   First Integrate the function over the interval:

$$\begin{aligned}\int_0^3 x^{1/2}\,dx &= \frac{2}{3}x^{3/2}\Big|_0^3\\ &= \frac{2}{3}(3^{3/2} - 0^{3/2})\\ &= \frac{2}{3}(3\sqrt{3}) = 2\sqrt{3}\end{aligned}$$

Now use the Mean-Value Theorem for Integrals formula:

$$\int_0^3 \sqrt{x}\ dx = f(x^*)(3-0) = 3\sqrt{x^*}$$

So we have the following equality to solve for x^*:

$$\begin{aligned} 3\sqrt{x^*} &= 2\sqrt{3} \\ 9x^* &= 4(3) \\ x^* &= \frac{12}{9} = \boxed{\frac{4}{3}} \end{aligned}$$

(b) $f(x) = \sin(x); \quad [-\pi,\ \pi]$

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(x)\ dx &= -\cos(x)|_{-\pi}^{\pi} \\ &= -(\cos(\pi) - \cos(-\pi)) \\ &= -((-1) - (-1)) = -(-1+1) = 0 \end{aligned}$$

$$\int_{-\pi}^{\pi} \sin(x)\ dx = f(x^*)(\pi - (-\pi)) = 2\pi\cos(x^*)$$

$$\begin{aligned} 2\pi\cos(x^*) &= 0 \\ \cos(x^*) &= 0 \\ x^* &= \cos^{-1}(0) \qquad -\pi \le x^* \le \pi \\ &\boxed{x^* = -\frac{\pi}{2},\ \frac{\pi}{2}} \end{aligned}$$

3. Evaluate the Integrals using Part 1 of the Fundamental Theorem of Calculus.

(a) $$\int_{-2}^{1} (x^2 - 6x + 12)\ dx$$

$$= \left(\frac{x^3}{3} - 6\frac{x^2}{2} + 12x\right)\Bigg|_{-2}^{1}$$
$$= \left(\frac{(1)^3}{3} - 3(1)^2 + 12(1)\right) - \left(\frac{(-2)^3}{3} - 3(-2)^2 + 12(-2)\right)$$
$$= \left(\frac{1}{3} - 3 + 12\right) - \left(-\frac{8}{3} - 12 - 24\right)$$
$$= \frac{1}{3} + 12 + \frac{8}{3} + 36$$
$$= 45 + \frac{9}{3} = 45 + 3 = \boxed{48}$$

(b) $$\int_{4}^{9} 2x\sqrt{x}\ dx$$

$$= 2\int_{4}^{9} x^{3/2}\ dx$$
$$= 2\left(\frac{2}{5}x^{5/2}\Bigg|_{4}^{9}\right)$$
$$= \frac{4}{5}\left(9^{5/2} - 4^{5/2}\right)$$
$$= \frac{4}{5}(3^5 - 2^5)$$
$$= \frac{4}{5}(243 - 32) = \frac{4}{5}(211) = \boxed{\frac{844}{5}}$$

(c) $$\int_{-\pi/4}^{\pi/4} \cos(x)\ dx$$

$$= \sin(x)|_{-\pi/4}^{\pi/4}$$
$$= \sin(\pi/4) - \sin(-\pi/4)$$
$$= \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) = \frac{2\sqrt{2}}{2} = \boxed{\sqrt{2}}$$

(d) $$\int_1^3 \frac{1}{x^2}\ dx = -\frac{1}{x}\bigg|_1^3 = -\left(\frac{1}{3} - \frac{1}{1}\right) = -\left(-\frac{2}{3}\right) = \boxed{\frac{2}{3}}$$

(e) $$\int_{-1}^{1} \sec^2(x)\ dx = \tan(x)|_{-1}^{1}$$
$$= \tan(1) - \tan(-1) = \tan(1) - (-\tan(1)) = \boxed{2\tan(1)}$$

(f) $$\int_{-1}^{1} |2x-1|\ dx$$

To evaluate this integral we must find out where the change in sign is for the function inside of the absolute value. This occurs at when $2x - 1 = 0$ or at $x = \frac{1}{2}$. So for $-1 \leq x \leq \frac{1}{2}$ we integrate $-(2x-1) = 1 - 2x$, and from $\frac{1}{2} \leq x \leq 1$ we integrate $(2x - 1)$. We can now use Theorem 4.5.5 to get:

$$\int_{-1}^{1} |2x-1|\ dx = \int_{-1}^{1/2} (1-2x)\ dx + \int_{1/2}^{1} (2x-1)\ dx$$
$$= (x - x^2)\big|_{-1}^{1/2} + (x^2 - x)\big|_{1/2}^{1}$$
$$= \left[\left(\frac{1}{2} - \frac{1}{4}\right) - ((-1) - (-1)^2)\right] + \left[(1-1) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$$
$$= \frac{1}{4} + 2 + 0 + \frac{1}{4}$$
$$= 2 + \frac{1}{2} = \boxed{\frac{5}{2}}$$

4. Define $F(x)$ by

$$F(x) = \int_1^x (3t^2 - 3)\ dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $\boxed{F'(x) = 3x^2 - 3}$

(b) Check the result by first integrating and then differentiating.

$$F(x) = \int_1^x (3t^2 - 3)\ dt = t^3 - 3t\Big|_1^x$$
$$= (x^3 - 3x) - (1 - 3) = x^3 - 3x + 2$$

$$\boxed{F'(x) = 3x^2 - 3}$$

5. Use Part 2 of the Fundamental Theorem of Calculus to find the derivatives.

(a) $$\frac{d}{dx}\int_1^x \sin(t^2)\ dt = \boxed{\sin(x^2)}$$

(b) $$\frac{d}{dx}\int_1^x \sqrt{1 - \cos(t)}\ dt = \boxed{\sqrt{1 - \cos(x)}}$$

(c) $$\frac{d}{dx}\int_x^0 t\sec(t)\ dt = -\frac{d}{dx}\int_0^x t\sec(t)\ dt = \boxed{-x\sec(x)}$$

(d) $$\frac{d}{du}\int_0^u |x|\ dx = \boxed{|u|}$$

6. *Let* $F(x) = \int_4^x \sqrt{t^2+9}\, dt$. *Find,*

(a) $$F(4) = \int_4^4 \sqrt{t^2+9}\, dt = \boxed{0}$$

(b) $$F'(4) = \sqrt{x^2+9}\Big|_{x=4} = \sqrt{16+9} = \sqrt{25} = \boxed{5}$$

(c) $$F''(4) = \frac{x}{\sqrt{x^2+9}}\Big|_{x=4} = \frac{4}{\sqrt{25}} = \boxed{\frac{4}{5}}$$