Rectilinear Motion Using Integration
Solutions To Selected Problems
Calculus 9^{th} Edition Anton, Bivens, Davis

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1. A particle moves along an s-axis. Use the given information to find the position function of the particle.

   (a) $v(t) = 3t^2 - 2t; \quad s(0) = 1$

$$s(t) = \int v(t)\ dt = \int 3t^2 - 2t\ dt = t^3 - t^2 + C$$

$$s(0) = 0^3 - 0^2 + C = 1$$
$$C = 1$$

$$\boxed{s(t) = t^3 - t^2 + 1}$$

   (b) $a(t) = 3\sin(3t); \quad v(0) = 3; \quad s(0) = 3$

$$v(t) = \int a(t)\ dt = \int 3\sin(3t)\ dt = -\cos(3t) + C$$

$$v(0) = -\cos(0) + C = 3$$
$$-1 + C = 3$$
$$C = 4$$
$$v(t) = -\cos(4t) + 4$$

$$s(t) = \int v(t)\ dt = \int (-\cos(3t) + 4)\ dt = -\frac{1}{3}\sin(3t) + 4t + C$$

$$s(0) = -\frac{1}{3}\sin(0) + 4(0) + C = 3$$
$$C = 3$$

$$\boxed{s(t) = -\frac{1}{3}\sin(3t) + 4t + 3}$$

(c) $a(t) = 2t^{-3}; \quad v(1) = 0; \quad s(1) = 0$

$$v(t) = \int 2t^{-3}\ dt = \frac{2t^{-2}}{-2} + C = -t^{-2} + C$$

$$\begin{aligned} v(1) &= -(-1)^{-2} + C = 0 \\ -1 + C &= 0 \\ C &= 1 \\ v(t) &= -t^{-2} + 1 \end{aligned}$$

$$s(t) = \int (-t^{-2} + 1)\ dt = \frac{-t^{-1}}{-1} + t + C = t^{-1} + t + C$$

$$\begin{aligned} s(1) &= 1^{-1} + 1 + C = 2 \\ 2 + C &= 2 \\ C &= 0 \end{aligned}$$

$$\boxed{s(t) = t^{-1} + t}$$

2. A particle moves with a velocity of $v(t)\ m/s$ along an s-axis. Find the displacement and the distance traveled by the particle during the given time interval.

   (a) $v(t) = \sin(t); \quad 0 \le t \le \frac{\pi}{2}$

   Total displacement over the time interval is given by:

$$\begin{aligned} \int_0^{\pi/2} v(t)\ dt &= \int_0^{\pi/2} \sin(t)\ dt \\ &= -\cos(t)\big|_0^{\pi/2} \\ &= -(\cos(\pi/2) - \cos(0)) = -(0 - 1) = \boxed{1} \end{aligned}$$

   Note that $\sin(t) \ge 0$ over this interval so that $|v(t)| = |\sin(t)| = \sin(t)$. Thus, the distance traveled is still $\int_0^{\pi/2} \sin(t)\ dt = \boxed{1}$

(b) $v(t) = \cos(t); \quad \frac{\pi}{2} \le t \le 2\pi$

Total displacement is given by:

$$\begin{aligned}\int_{\pi/2}^{2\pi} \cos(t)\ dt &= \sin(t)\Big|_{\pi/2}^{2\pi} \\ &= \sin(2\pi) - \sin(\pi/2) = 0 - 1 = \boxed{-1}\end{aligned}$$

Now $|v(t)| = |\cos(t)|$ is both positive and negative over this interval. (Draw a graph of $\cos(t)$ to help see this).

$$|\cos(t)| = \begin{cases} -\cos(t) & \frac{\pi}{2} \le t \le \frac{3\pi}{2} \\ \cos(t) & \frac{3\pi}{2} \le t \le 2\pi \end{cases}$$

So the total distance traveled is given by:

$$\begin{aligned}\int_{\pi/2}^{2\pi} |\cos(t)|\ dt &= \int_{\pi/2}^{3\pi/2} (-\cos(t))\ dt + \int_{3\pi/2}^{2\pi} \cos(t)\ dt \\ &= -\sin(t)\Big|_{\pi/2}^{3\pi/2} \quad + \quad \sin(t)\Big|_{3\pi/2}^{2\pi} \\ &= -\left(\sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right) + \left(\sin(2\pi) - \sin\left(\frac{3\pi}{2}\right)\right) \\ &= -(-1-1) + (0 - (-1)) \\ &= -(-2) + 1 = 2 + 1 = \boxed{3}\end{aligned}$$

3. A particle moves along with acceleration $a(t)\ m/s^2$ along an s-axis and has velocity $v_0\ m/s$ at time $t = 0$. Find the displacement and the distance traveled by the particle during the given time interval.

$$a(t) = 3; \quad v_0 = -1; \quad 0 \leq t \leq 2$$

Note that this is constant acceleration, so we automatically know that

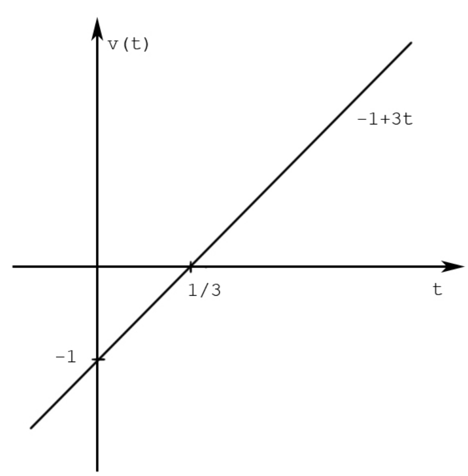
$$v(t) = -1 + 3t$$

Now the total displacement is given by:

$$\int_0^2 (-1+3t)\ dt = -t + \frac{3}{2}t^2 \Big|_0^2$$

$$= \left(-2 + \frac{3}{2}(2)^2\right) - \left(-0 + \frac{3}{2}0^2\right)$$
$$= (-2 + 3(2)) - 0 = -2 + 6 = \boxed{4}$$

Now we must analyze $|v(t)| = |-1+3t|$, where $-1+3t$ is a line which is zero at $t = \frac{1}{3}$



$$|-1+3t| = \begin{cases} -(-1+3t) = 1-3t & 0 \le t \le \frac{1}{3} \\ -1+3t & \frac{1}{3} \le t \le 2 \end{cases}$$

So the Total distance traveled is given by:

$$\int_0^2 |-1+3t|\ dt = \int_0^{1/3} (1-3t)\ dt + \int_{1/3}^2 (-1+3t)\ dt$$

$$= \left(t - \frac{3}{2}t^2 \Big|_0^{1/3} \right) + \left(-t + \frac{3}{2}t^2 \Big|_{1/3}^{2} \right)$$

$$= \left(\frac{1}{3} - \frac{3}{2}\left(\frac{1}{3}\right)^2 \right) + \left(-2 + \frac{3}{2}(2)^2 \right) - \left(-\frac{1}{3} + \frac{3}{2}\left(\frac{1}{3}\right)^2 \right)$$

$$= \frac{1}{3} - \frac{3}{2} \cdot \frac{1}{9} - 2 + 6 + \frac{1}{3} - \frac{1}{6}$$

$$= \frac{2}{3} - \frac{2}{6} + 4$$

$$= \frac{1}{3} + 4 = \frac{1+12}{3} = \boxed{\frac{13}{3}}$$

4. In each part, use the given information to find the position, velocity, speed, and acceleration at time $t = 1$.

(a)

$$v(t) = \sin(\pi t/2) \quad s = 0.\ when\ t = 0$$
$$v(1) = \sin(\pi/2) = 1$$
$$\boxed{v(1) = 1}$$

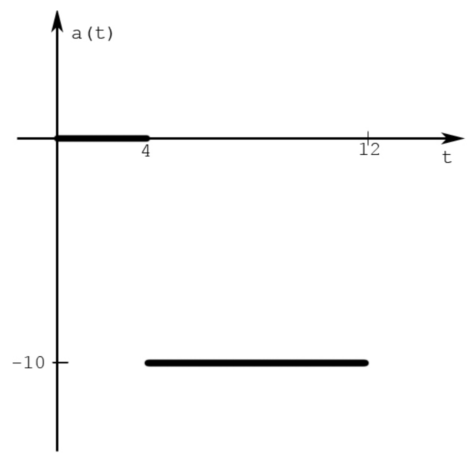
$$a(t) = v'(t) = \frac{\pi}{2}\cos(\pi t/2)$$
$$a(1) = \frac{\pi}{2}\cos(\pi/2) = \frac{\pi}{2}(0) = 0$$
$$\boxed{a(1) = 0}$$

$$s(t) = \int \sin(\pi t/2)\ dt = -\frac{2}{\pi}\cos(\pi t/2) + C$$
$$s(0) = -\frac{2}{\pi}\cos(0) + C = 0$$
$$C = \frac{2}{\pi}$$

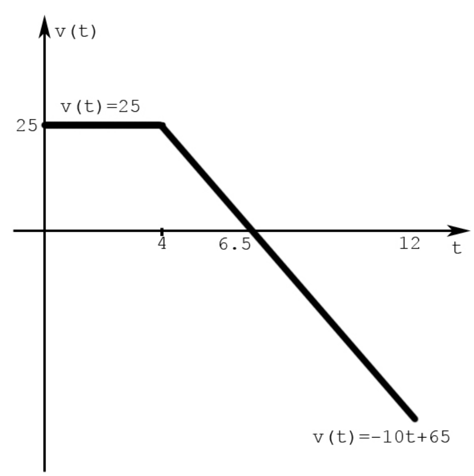
$$s(t) = -\frac{2}{\pi}\cos(\pi t/2) + \frac{2}{\pi}$$
$$s(1) = -\frac{2}{\pi}\cos(\pi/2) + \frac{2}{\pi} = 0 + \frac{2}{\pi} = \frac{2}{\pi}$$
$$\boxed{s(1) = \frac{2}{\pi}}$$

5. Suppose at time $t = 0$ a particle is at the origin of an x-axis and has a velocity of $v_0 = 25\ cm/s$. For the first 4 seconds thereafter it has no acceleration, and then it is acted on by a retarding force that produces a constant negative acceleration of $a = -10\ cm/s^2$.

(a) Sketch the acceleration versus time curve over the interval $0 \leq t \leq 12$.



(b) Sketch the velocity versus time curve over the time interval $0 \leq t \leq 12$.



To determine the line $-10t+65$ in part (b) above, we know that the slope at $t=4$ is -10 as the acceleration there is -10. We also need the point at $t=4$. Since the velocity is always 25 for the first four seconds, we have the point (4,25). Using the point-slope formula we get:

$$
\begin{aligned}
y - 25 &= -10(x-4) \\
y &= -10x + 40 + 25 \\
y &= -10x + 65 \quad for\ t \geq 4
\end{aligned}
$$

(c) Find the x-coordinate of the particle at times $t = 8$ seconds and $t = 12$.

First find the function $s(t) = \int v(t)\,dt$, where $v(t) = \begin{cases} 25 & 0 \leq t \leq 4 \\ -10t+25 & 4 \leq t \leq 12 \end{cases}$

$$
\begin{aligned}
s(t) &= \int 25\ dt = 25t + C \\
s(0) &= 25(0) + C = 0 \\
C &= 0 \\
s(t) &= 25t \quad 0 \leq t \leq 4
\end{aligned}
$$

$$
\begin{aligned}
s(t) &\int 65 - 10t\ dt = 65t - 5t^2 + C \\
s(4) &= 100 \ \text{ from the above formula, } s(t) = 25t \\
65(4) - 5(4)^2 + C &= 100 \\
260 - 5(16) + C &= 100 \\
180 + C &= 100 \\
C &= -80
\end{aligned}
$$

$$s(t) = 64t - 5t^2 - 80 \quad 4 \leq t \leq 12$$

Now we can find $s(8)$ and $s(12)$:

$$\begin{aligned} s(8) &= 65(8) - 5(8)^2 - 80 \\ &= 520 - 320 - 80 = 200 - 80 = 120 \end{aligned}$$

$$\boxed{s(8) = 120}$$

$$\begin{aligned} s(12) &= 65(12) - 5(12)^2 - 80 \\ &= 780 - 720 - 80 = 60 - 80 = -20 \end{aligned}$$

$$\boxed{s(12) = -20}$$

(d) What is the maximum x-coordinate of the particle over the time interval $0 \leq t \leq 12$? This occurs when

$$\begin{aligned} s'(t) &= v(t) = 0 \\ \rightarrow\ 65 - 10t &= 0 \\ 10t &= 65 \\ t &= 65/10 = \boxed{6.5} \end{aligned}$$

6. Spotting a police car, you hit the brakes on your new Porsche to reduce your speed to from 90 mi/h to 60 mi/h at a constant rate over a distance of 200 feet. (Note that 88 ft/sec = 60 mi/h. So 1 mi/h = 22/15 ft/sec).

   (a) Find the acceleration in ft/s^2.

   We need to first find the v(t) and s(t) functions.

$$a(t) = a_0 \quad (constant)$$

$$v(t) = \int a_0 \; dt = a_0 t + v_0$$
$$v_0 = 90\frac{mi}{h} = 90\left(\frac{22}{15}\frac{ft}{sec}\right)$$
$$= 6(22)\frac{ft}{sec} = 132\frac{ft}{sec}$$

$$v(t) = a_0 t + 132$$

$$s(t) = \int v(t) \; dt = \int a_0 t + 132 \; dt$$
$$= a_0 \frac{t^2}{2} + 132t + s_0$$
$$s_0 = 0 \quad (given)$$

$$s(t) = \frac{1}{2}a_0 t^2 + 132t$$

   Now we use the fact that at $s = 200$, we have

$$v(t) = 60mi/h = 88ft/sec$$
$$\rightarrow 88 = a_0 t + 132$$
$$-44 = a_0 t$$
$$-\frac{44}{a_0} = t$$

Now substitute this value of t into the equation $s(t) = 200 = \frac{1}{2}a_0t^2 + 132$

$$\begin{aligned}
200 &= \frac{1}{2}a_0\left(\frac{-44}{a_0}\right)^2 + 132\left(\frac{-44}{a_0}\right) \\
200 &= \frac{a_0}{2}\left(\frac{1936}{a_0^2}\right) - \frac{5808}{a_0} \\
200 &= \frac{968}{a_0} - \frac{5808}{a_0} \\
200 &= -\frac{4840}{a_0} \\
200a_0 &= -4840 \\
a_0 &= -\frac{4840}{200} \\
&\boxed{a(t) = a_0 = -\frac{121}{5}}
\end{aligned}$$

$$\begin{aligned}
\to v(t) &= -\frac{121}{5}t + 132 \\
\to s(t) &= -\frac{121}{5}\frac{t^2}{2} + 132t = -\frac{121}{10}t^2 + 132t = -12.1t^2 + 132t
\end{aligned}$$

(b) How long does it take for you to reduce your speed to 55 mi/h?
Since our equations are in ft/sec, convert $55\frac{mi}{h} = 55\left(\frac{22}{15}\frac{ft}{sec}\right) = \frac{242}{3}\frac{ft}{sec}$.
Now solve using the velocity function:

$$\begin{aligned}
\frac{242}{3} &= -\frac{121}{5}t + 132 \\
\frac{242}{3} - 132 &= -\frac{121}{5}t \\
\frac{242 - 396}{3} &= -\frac{121}{5}t \\
\left(-\frac{154}{3}\right)\left(-\frac{5}{121}\right) &= t \\
\frac{(2)(7)(11)(5)}{(3)(11)(11)} &= t \\
&\boxed{t = \frac{70}{33}}
\end{aligned}$$

(c) At the acceleration obtained in part (a), how long would it take for you to bring your Porsche to a complete stop from 90 mi/h ?

We are asked to find at what time t is $v(t) = 0$?

$$-\frac{121}{5}t + 132 = 0$$
$$-\frac{121}{5}t = -132$$
$$t = 132 \cdot \frac{5}{121}$$
$$t = \frac{(11)(12)(5)}{(11)(11)}$$
$$\boxed{t = \frac{60}{11}\ sec\ \approx 5.45\ sec}$$

7. A projectile is launched vertically upward from ground level with an initial velocity of $v_0 = 112\ ft/s$.

(a) Find the velocity at $t = 3$ and $t = 5$.

Take gravity to be the acceleration $a = -32\ ft/s^2$ so we have

$$v(t) = v_0 + at = 112 - 32t$$
$$v(3) = 112 - 32(3) = 112 - 96 = \boxed{16\ ft/s}$$
$$v(5) = 112 - 32(5) = 112 - 160 = \boxed{-48\ ft/s}$$

(b) How high will the projectile rise?

Solve $v(t) = 0$ for t to find the time when the particle "stops" rising to turn around and fall back. This will the time at which the particle is at its highest.

$$\begin{aligned}112 - 32t &= 0\\ 112 &= 32t\\ t &= \frac{112}{32} = \frac{7}{2}\end{aligned}$$

Now we need the actual position function $s(t)$ and then find $s(7/2)$:

$$\begin{aligned}s(t) &= \int v(t)\ dt = \int 112 - 32t\ dt = 112t - 16t^2 + s_0 \quad (s_0 = 0)\\ s(t) &= 112t - 16t^2\end{aligned}$$

$$\begin{aligned}s(7/2) &= 112\frac{7}{2} - 16\left(\frac{7}{2}\right)^2\\ &= 56(7) - 16\frac{49}{4}\\ &= 392 - 196 = \boxed{196\ ft}\end{aligned}$$

(c) Find the <u>speed</u> of the projectile when it hits the ground.

Find when $s(t) = 0$ for t to find out the time when it hits the ground. Then find $|v(t)|$ at this time to find its speed upon impact.

$$\begin{aligned}-16t^2 + 112t &= 0\\ -t(16t - 112) &= 0\\ t &= 0 \quad \text{Initial starting time, disregard}\\ 6t - 112 &= 0\\ 16t &= 112\\ t &= 112/16 = 7\end{aligned}$$

$$v(7) = -32(7) + 112 = -224 + 112 = -112$$

So the speed upon impact is $\boxed{112ft/s}$