Average Value of a function
Solutions To Selected Problems
Calculus 9^{th} Edition Anton, Bivens, Davis

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1. (a) Find f_{ave} of $f(x) = 2x$ over $[0, 4]$.

$$\begin{aligned} f_{ave} &= \frac{1}{4-0} \int_0^4 2x \, dx \\ &= \frac{1}{4} \, x^2\Big|_0^4 \\ &= \frac{1}{4}(4^2 - 0^2) \\ &= \frac{1}{4}(16) = \boxed{4} \end{aligned}$$

(b) Find a point x^* in $[0, 4]$ such that $f(x^*) = f_{ave}$

$$\begin{aligned} f(x^*) &= 2x^* \\ \text{Want } 2x^* &= 4 \\ &\boxed{x^* = 2} \end{aligned}$$

2. Let $f(x) = 3x^2$

(a) Find the arithmetic average of the values $f(0.4), f(0.8), f(1.2), f(1.6)$, and $f(2.0)$.

$$\begin{aligned} n &= 5 \\ \bar{a} &= \frac{1}{5}(f(0.4) + f(0.8) + f(1.2) + f(1.6) + f(2.0)) \\ &= \frac{1}{5}(0.48 + 1.92 + 4.32 + 7.68 + 12) \\ &= \frac{1}{5}(26.4) = \boxed{5.28} \end{aligned}$$

(b) Find the average value of f on $[0, 2]$

$$\begin{aligned} f_{ave} &= \frac{1}{2-0} \int_0^2 3x^2 \, dx \\ &= \frac{1}{2} \, x^3\Big|_0^2 = \frac{1}{2}(2^3 - 0^3) = \frac{1}{2}(8) = \boxed{4} \end{aligned}$$

3. Suppose that f is a linear function. Using the graph of f, explain why the average value of f on $[a, b]$ is $f\left(\frac{a+b}{2}\right)$.

Note that if f is a linear function, then $f(c_1x_1 + c_2x_2) = c_1f(x_1) + c_2f(x_2)$, for constants $c_1,\ c_2$. So that means

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{2}a + \frac{1}{2}b\right) = \frac{1}{2}f(a) + \frac{1}{2}f(b) = \frac{1}{2}(f(a) + f(b))$$

What this is saying is that it takes the average value of the end points. Since it is linear, it is precisely the average value of the function as $(a+b)/2$ is halfway between the points $[a, b]$.

4. (a) Suppose that the velocity function of a particle moving a long a coordinate line is $v(t) = 3t^3 + 2$. Find the average velocity of the particle over the time interval $1 \leq t \leq 4$ by integrating.

$$\begin{aligned} v_{ave} &= \frac{1}{4-1}\int_1^4 3t^2 + 2\ dt \\ &= \frac{1}{3}\left(\frac{3}{4}t^4 + 2t\Big|_1^4\right) \\ &= \frac{1}{3}\left[\left(\frac{3}{4}(4^4) + 2(4)\right) - \left(\frac{3}{4}(1)^4 + 2(1)\right)\right] \\ &= \frac{1}{3}\left[192 + 8 - \frac{3}{4} - 2\right] \\ &= \frac{1}{3}\left[198 - \frac{3}{4}\right] = \frac{1}{3}\left[\frac{792-2}{4}\right] = \frac{1}{12}(789) = \boxed{\frac{263}{4}} \end{aligned}$$

(b) Suppose that the position function of a particle moving along a coordinate line is $s(t) = 6t^2 + t$. Find the average velocity of the particle over the time interval $1 \leq t \leq 4$ algebraically.

$$\begin{aligned} \frac{s(4) - s(1)}{4-1} &= \frac{(6(4)^2 + 4) - (6(1)^2 + 1)}{3} \\ &= \frac{100 - 7}{3} = \frac{93}{3} = \boxed{31} \end{aligned}$$

5. Water is run at a constant rate of 1 ft^3/min to fill a cylindrical tank of radius 3 ft and height 5 ft. Assuming that the tank is initially empty, make a conjecture about the average weight of the water in the tank over the time period required to fill it, and check your conjecture by integrating. [Take the weight density of water to be 62.4 lb/ft^3].

The total volume of the tank is $V = \pi r^2 h = \pi 9(5) = 45\pi$.

The time to fill the tank is $t = \frac{Volume}{Rate} = \frac{45\pi}{1} = 45\pi$. Thus the average weight should occur when the tank is half way full, which is at time $t = \frac{45\pi}{2}$, with the average weight being $64.2\left(\frac{45\pi}{2}\right) = 32.1(45\pi) = 1401\pi$.

We can check this by integrating $62.4t$ from $t = 0$ to $t = 45\pi$.

$$\begin{aligned} W_{ave} &= \frac{1}{45\pi - 0}\int_0^{45\pi} 62.4t\ dt \\ &= \frac{1}{45\pi}\left[\frac{62.4}{2}t^2\Big|_0^{45\pi}\right] \\ &= \frac{31.2}{45\pi}\left[(45\pi)^2 - 0\right] \\ &= 31.2(45\pi) = \boxed{1404\pi} \end{aligned}$$

6. Find a positive value of k such that the average value of $f(x) = \sqrt{3x}$ over the interval $[0, k]$ is 6.
We need to solve the following for k:

$$\begin{aligned} \frac{1}{k-0}\int_0^k \sqrt{3x}\ dx &= 6 \\ \frac{1}{k}\int_0^k \sqrt{3x}\ dx &= 6 \\ \int_0^k \sqrt{3x}\ dx &= 6k \end{aligned}$$

To help evaluate, make a u-substitution.

$$\begin{aligned} let\ u &= 3x \\ du &= 3\ dx \\ \frac{1}{3}du &= dx \end{aligned}$$

$$\frac{1}{3}\int_0^k u^{1/2}\ du = 6k$$

$$\frac{1}{3}\left(\frac{2}{3}\ u^{3/2}\Big|_0^k\right) = 6k$$

$$\frac{2}{9}\ (3x)^{3/2}\Big|_0^k = 6k$$

$$\begin{aligned} (3k)^{3/2} - 0 &= 6k\frac{9}{2} \\ 3\sqrt{3}k^{3/2} &= 27k \\ \frac{k^{3/2}}{k} &= 27\frac{1}{3\sqrt{3}} \\ k^{1/2} &= \frac{9}{\sqrt{3}} \\ k &= \left(\frac{9}{\sqrt{3}}\right)^2 = \frac{81}{3} = \boxed{27} \end{aligned}$$