Evaluating Definite Integrals by Substitution
Solutions To Selected Problems
Calculus 9^{th} Edition Anton, Bivens, Davis

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1. Express the integral in terms of the variable u, but do not evaluate it.

(a) $$\int_1^3 (2x-1)^3\ dx; \quad u = 2x-1$$

$$du = 2\ dx$$

$$\frac{1}{2}\ du = dx$$

$$x = 1 \to u = 2(1) - 1 = 1$$
$$x = 3 \to u = 2(3) - 1 = 5$$

$$\int_1^3 (2x-1)^3\ dx = \boxed{\frac{1}{2}\int_0^5 u^3\ du}$$

(b) $$\int_0^4 3x\sqrt{25-x^2}\ dx; \quad u = 25 - x^2$$

$$du = -2x\ dx$$

$$-\frac{1}{2}\ du = x\ dx$$

$$x = 0 \to u = 25 - 0^2 = 25$$
$$x = 4 \to u = 25 - 16 = 9$$

$$\int_0^4 3x\sqrt{25-x^2}\ dx = -\frac{1}{2}\int_{25}^9 3\sqrt{u}\ du = \boxed{\frac{3}{2}\int_9^{25} \sqrt{u}\ du}$$

2. Evaluate the definite integral by making an appropriate u-substitution.

(a) $$\int_0^{\pi/2} 4\sin(x/2)\ dx; \quad \textit{let } u = \frac{x}{2}$$

$$du = \frac{1}{2}\ dx$$

$$2\ du = dx$$

$$x = 0 \to u = 0/2 = 0$$

$$x = \pi/2 \to u = \pi/4$$

$$\begin{aligned}\int_0^{\pi/2} 4\sin(x/2)\ dx &= 2(4)\int_0^{\pi/4} \sin(u)\ du \\ &= 8\left[-\cos(u)|_0^{\pi/4}\right] \\ &= -8[\cos(\pi/4) - \cos(0)] \\ &= -8\left[\frac{\sqrt{2}}{2} - 1\right] = \boxed{8 - 4\sqrt{2}}\end{aligned}$$

(b) $$\int_{-2}^{-1} \frac{x}{(x^2+2)^2}\ dx; \quad \textit{let } u = x^2 + 2$$

$$du = 2x\ dx$$

$$\frac{1}{2}\ du = dx$$

$$x = -2 \to u = 4 + 2 = 6$$

$$x = -1 \to u = 1 + 2 = 3$$

$$\int_{-2}^{-1} \frac{x}{(x^2+2)^2}\,dx = \frac{1}{2}\int_6^3 \frac{1}{u^3}\,du = -\frac{1}{2}\int_3^6 u^{-3}\,du$$

$$= -\frac{1}{2}\left(\frac{u^{-2}}{-2}\Big|_3^6\right) = \frac{1}{4}\left[\frac{1}{6^2} - \frac{1}{3^2}\right]$$

$$= \frac{1}{4}\left[\frac{1}{36} - \frac{1}{9}\right] = \frac{1}{4}\left[\frac{1-4}{36}\right]$$

$$= \frac{-3}{4(36)} = -\frac{1}{4(12)} = \boxed{-\frac{1}{48}}$$

3. Evaluate the definite integral by expressing it in terms of u and evaluating the resulting integral using a formula from geometry.

(a) $\displaystyle\int_{-5/3}^{5/3} \sqrt{25-9x^2}\,dx;\quad u = 3x$

$$du = 3\,dx$$

$$\frac{1}{3}\,du = dx$$

$$x = -5/3 \to u = -5$$
$$x = 5/3 \to u = 5$$

$$\int_{-5/3}^{5/3} \sqrt{25-9x^2}\,dx = \int_{-5/3}^{5/3} \sqrt{25-(3x)^2}\,dx = \frac{1}{3}\int_{-5}^{5} \sqrt{25-u^2}\,du$$

Now let $y = \sqrt{25-u^2}$. Note that this is the same as $y^2+u^2 = 25$, which is the top half of a circle centered at the origin with a radius of 5. The area of half of the circle is $\frac{1}{2}\pi r^2 = \frac{25}{2}\pi$. So the total integral is $\frac{1}{3}\left(\frac{25}{2}\pi\right) = \boxed{\frac{25}{6}\pi}$

4. Evaluate the integrals by any method.

(a)

$$\int_1^3 \frac{x+2}{\sqrt{x^2+4x+7}}\,dx; \quad \textit{let } u = x^2+4x+7$$

$$du = 2x+4\ dx$$

$$\frac{1}{2}\ du = (x+2)\ du$$

$$x = 1 \to u = 1+4+7 = 12$$
$$x = 3 \to u = 3^2+4(3)+7 = 28$$

$$\int_1^3 \frac{x+2}{\sqrt{x^2+4x+7}}\,dx = \frac{1}{2}\int_{12}^{28} \frac{du}{\sqrt{u}} = \frac{1}{2}\left(2u^{1/2}\Big|_{12}^{28}\right)$$
$$= \sqrt{28}-\sqrt{12} = 2\sqrt{7}-2\sqrt{3} = \boxed{2\left(\sqrt{7}-\sqrt{3}\right)}$$

(b)

$$\int_0^{\pi/4} 4\sin(x)\cos(x)\ dx \quad \textit{let } u = \sin(x)$$

$$du = \cos(x)\ dx$$

$$x = 0 \to u = \sin(0) = 0$$
$$x = \pi/4 \to u = \sin(\pi/4) = \frac{\sqrt{2}}{2}$$

$$\int_0^{\pi/4} 4\sin(x)\cos(x)\ dx = 4\int_0^{\sqrt{2}/2} u\ du = 4\left(\frac{1}{2}u^2\Big|_0^{\sqrt{2}/2}\right)$$
$$= 2\left[\left(\frac{\sqrt{2}}{2}\right)^2 - 0^2\right] = 2\left(\frac{2}{4}\right) = \boxed{1}$$

(c) $$\int_0^1 \frac{y^2}{\sqrt{4-3y}}\,dy \quad let\ u = 4-3y$$

$$du = -3\ dy$$

$$-\frac{1}{3}\ du = dy$$

$$y = 0 \rightarrow u = 4$$
$$y = 1 \rightarrow u = 1$$

Need to solve for y, then find y^2

$$u - 4 = -3y$$
$$-\frac{1}{3}(u-4) = y$$
$$y^2 = \frac{1}{9}(u^2 - 8u + 16)$$

$$\int_0^1 \frac{y^2}{\sqrt{4-3y}}\,dy = -\frac{1}{3}\int_4^1 \frac{\frac{1}{9}(u^2-8u+16)}{\sqrt{u}}\ du$$

$$= \frac{1}{27}\int_1^4 \frac{u^2}{\sqrt{u}} - \frac{8u}{\sqrt{u}} + \frac{16}{\sqrt{u}}\ du$$

$$= \frac{1}{27}\int_1^4 u^{3/2} - 8u^{1/2} + 16u^{-1/2}\ du$$

$$= \frac{1}{27}\left[\frac{2}{5}u^{5/2} - 8\left(\frac{2}{3}u^{3/2}\right) + 16(2u^{1/2})\Big|_1^4\right]$$

$$= \frac{1}{27}\left[\left(\frac{2}{5}(4)^{5/2} - \frac{16}{3}(4)^{3/2} + 32(4)^{1/2}\right) - \left(\frac{2}{5}(1) - \frac{16}{3}(1) + 32(1)\right)\right]$$

$$= \frac{1}{27}\left[\left(\frac{2}{5}(2)^5 - \frac{2^4}{3}(2)^3 + 2^5(2)\right) - \left(\frac{2}{5} - \frac{2^4}{3} + 2^5\right)\right]$$

$$= \frac{1}{27}\left[\left(\frac{2^6(3) - 2^7(5) + 2^6(15)}{15}\right) - \left(\frac{2(3) - 2^4(5) + 2^5(15)}{15}\right)\right]$$

$$= \frac{1}{27} \cdot \frac{1}{15}\left[3(2^6 - 2) - 5(2^7 - 2^4) + 15(2^6 - 2^5)\right]$$

$$= \frac{1}{405}\left[3(62) - 5(112) + 15(32)\right]$$

$$= \frac{1}{405}\left[186 - 560 - 480\right] = \boxed{\frac{106}{405}}$$