8.2 SEPARATION OF VARIABLES 06/21

CONSIDER $y' + y = 2e^x$

$f'(x) + f(x) = 2e^x$

THE SOLUTION TO THIS DIFFERENTIAL EQUATION IS $\underline{y = e^x}$ OR $\underline{f(x) = e^x}$

$y' + y = 2e^x$ IS A FIRST-ORDER DIFFERENTIAL EQUATION (DE)

$y'' + y' + y = e^x$ IS A SECOND-ORDER DE

SOME DE'S ARE SEPERABLE. THIS MEANS THE y TERMS CAN BE ISOLATED W/ dy ON ONE SIDE OF THE EQUATION.

EX1) THIS IS A SEPERABLE DE : $(1+x)\frac{dy}{dx} = y$

$(1+x)\,dy = y\,dx$

$\frac{1}{y}\,dy = \frac{1}{1+x}\,dx$

WE CAN SOLVE A SEPERABLE DE BY ISOLATING THE y AND dy TERMS TO ONE-SIDE OF THE EQUATION AND THEN INTEGRATE BOTH SIDES

$$\int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx$$

$$\ln|y| = \ln|1+x| + C$$

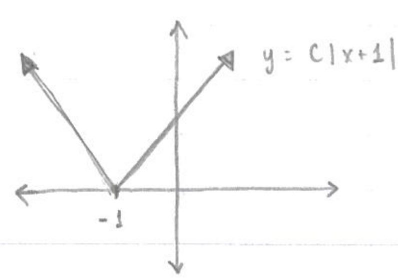
e e

$$e^{\ln|y|} = e^{\ln|1+x|} \cdot e^{C} \leftarrow \text{CONSTANT}$$

$$\boxed{y = C\,|1+x|}$$

NOTE: $y = C|x+1| = \begin{cases} C(x+1), & x \geq - \\ C(-x-1), & x < - \end{cases}$

THIS IS A GENERAL SOLUTION TO ONE DE.



NOW LET'S VERIFY THAT OUR SOLUTION IS CORRECT!

FOR $x \geq -1$

$y = C(1+x)$

$y' = C$

FOR $x < -1$

$y = C(-1-x)$

$y' = -C$

$(1+x) \cdot y' = y$ ← ORIGINAL DE → $(1+x) \cdot -C = C(-1-x)$

$(1+x) \cdot C = C(1+x)$ ✓

$(1+x) \cdot -C = -C(1+x)$ ✓

AN INITIAL VALUE PROBLEM IS A DE WHERE YOU ARE GIVEN AN ORDERED PAIR SUCH THAT YOUR GENERAL SOLUTION MUST PASS THROUGH THIS GIVEN ORDERED PAIR

THIS BASICALLY MEANS YOU CAN SOLVE FOR C.

SOLVE THE INITIAL VALUE PROBLEM.

$y' - 2y = 0 \;;\; y(0) = 2$

$\frac{dy}{dx} = 2y$

$dy = 2y\,dx$

$\frac{1}{y}\,dy = 2\,dx$

$\int \frac{1}{y}\,dy = \int 2\,dx$

$= \ln|y| = 2x + C$ ← $e^{2x+C} = e^{2x} \cdot \underbrace{e^{C}}_{\text{CONSTANT}}$

$e \qquad e$

$y = Ce^{2x}$

USING $(0, 2)$

$y = Ce^{2x}$

$2 = Ce^{2(0)}$

$2 = C$

$$\boxed{\begin{array}{l}\text{OUR EXACT SOLUTION IS} \\ \quad y = 2e^{2x}\end{array}}$$

EX 2) SOLVE

$$y' = \frac{y^2 - y}{\sin x}$$

$$\frac{dy}{dx} = \frac{y^2 - y}{\sin x}$$

$$\frac{1}{y^2 - y}\, dy = \csc x \, dx$$

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$$

$$1 = A(y-1) + By$$

$y = 1 : 1 = B$ $\quad$ $x = 0 : 1 = -A$

$A = -1$

$$\int \frac{1}{y(y-1)}\, dy = \int \csc x \, dx$$

$$\int \frac{-1}{y} + \frac{1}{y-1}\, dy = \int \csc x \, dx$$

$$-\ln|y| + \ln|y-1| = \ln|\csc x - \cot x| + C$$

$$e^{\ln\left|\frac{y-1}{y}\right|} = e^{\ln|\csc x - \cot x| + C}$$

$$\frac{y-1}{y} = C|\csc x - \cot x|$$

$$1 - \frac{1}{y} = C|\csc x - \cot x|$$

$$-\frac{1}{y} = C|\csc x - \cot x| - 1$$

$$\frac{1}{y} = C|\csc x - \cot x| + 1$$

$$\boxed{y = \frac{1}{C|\csc x - \cot x| + 1}}$$