8.4 FIRST-ORDER DIFFERENTIAL EQUATIONS AND APPLICATIONS 06/21

WHAT IF THE DE IS NOT SEPERABLE?

WE WILL ONLY DEAL W/ NON-SEPERABLE DE's THAT CAN BE SOLVED USING AN "INTEGRATING FACTOR" DEFINED TO BE $\mu(x) = e^{\int p(x)\,dx}$

THIS METHOD WORKS ON DE's OF THE FORM

$$* \quad \boxed{\frac{dy}{dx} + p(x)\,y = q(x)}$$

TO SOLVE $\frac{dy}{dx} + p(x)\,y = q(x)$ WE USE $\mu(x) = e^{\int p(x)\,dx}$

INTEGRATING FACTOR: $\mu = e^{\int p(x)\,dx}$

$$\frac{d\mu}{dx} = \underbrace{e^{\int p(x)\,dx}}_{\mu} \cdot p(x)$$

$$* \quad \frac{d\mu}{dx} = \mu\, p(x)$$

NOW WE MULTIPLY BOTH SIDES OF OUR DE BY μ

$$\frac{dy}{dx} + p(x)\,y = q(x)$$

$$\mu \cdot \frac{dy}{dx} + \underbrace{\mu\, p(x)}_{\frac{d\mu}{dx}}\,y = \mu\, q(x)$$

$$\underbrace{\mu \cdot \frac{dy}{dx} + \frac{d\mu}{dx}\,y}_{} = \mu\, q(x)$$

$$\frac{d}{dx}\left[\mu \cdot y\right] = \mu\, q(x)$$

NOW, WE INTEGRATE BOTH SIDES!

$$\int \frac{d}{dx} \mu \cdot y \, dx = \int \mu \, q(x) \, dx$$

$$\mu \cdot y = \int \mu \, q(x) \, dx$$

$$\boxed{y = \frac{1}{\mu} \cdot \int \mu \, q(x) \, dx}$$

EX1) SOLVE USING THE INTEGRATING FACTOR METHOD.

$\frac{dy}{dx} - y = e^{2x}$

$p(x) = -1$

$q(x) = e^{2x}$

$\mu = e^{\int p(x)\,dx}$

$$\mu = e^{\int (-1)\,dx} = e^{-x}$$

$$y = \frac{1}{\mu} \int \mu \, q(x) \, dx$$

$$y = \frac{1}{e^{-x}} \int e^{-x} \cdot e^{2x} \, dx$$

$$= e^{x} \int e^{x} \, dx$$

$$y = e^{x} [e^{x} + C]$$

$$\boxed{y = e^{2x} + Ce^{x}}$$

* EXAM TYPE PROBLEM WILL MOST LIKELY REQUIRE INTEGRATING FACTOR

EX 2) SOLVE THE INITIAL VALUE PROBLEM

$y' - \frac{3}{x}y = x^2 \qquad x > e \ \& \ y(e) = 0$

$p(x) = -\frac{3}{x}$

$q(x) = x^2$

$\mu = e^{\int (-\frac{3}{x})\,dx} = e^{-3\ln x} = e^{\ln x^{-3}} = e^{\ln \frac{1}{x^3}} = \frac{1}{x^3}$

$y = \frac{1}{\frac{1}{x^3}} \int \frac{1}{x^3} \cdot x^2 \, dx$

$= x^3 \int \frac{1}{x}\,dx$

$= x^3[\ln|x| + C]$

$= x^3 \ln x + Cx^3$

USING $(e, 0)$

$0 = e^3 \ln e + Ce^3$

$0 = e^3 + Ce^3$

$0 = 1 + C$

$C = -1$

$\boxed{y = x^3 \ln x - x^3}$

EX 3) SOLVE THE INITIAL VALUE PROBLEM

$\frac{dy}{dx} - 2xy = 2x \qquad y(0) = 3$

$p(x) = -2x$

$q(x) = 2x$

$\mu = e^{\int -2x\,dx} = e^{-x^2}$

$y = \frac{1}{\frac{1}{e^{x^2}}} \int e^{-x^2} \cdot 2x \, dx$

$y = e^{x^2} \int e^{-x^2} \cdot 2x \, dx$

LET $u = -x^2$

$du = -2x\,dx$

$-du = 2x\,dx$

$y = e^{x^2}\left[-\int e^u \, du\right]$

$y = e^{x^2}[-e^u + C]$

$y = e^{x^2}[-e^{-x^2} + C]$

$y = -1 + Ce^{x^2}$

USING $(0, 3)$

$3 = -1 + Ce^{(0)^2}$

$3 = -1 + C$

$C = 4$

$\boxed{y = -1 + 4e^{x^2}}$

EX 4) $y' + (\ln x)y = x^{-x} \qquad x \geq 1 \ \& \ y(2) = 0$

$p(x) = \ln x$

$q(x) = x^{-x}$

$$\mu = e^{\int \ln x \, dx} = e^{x \ln x - x} = e^{\ln x^x - x} = \frac{e^{\ln x^x}}{e^x} = \frac{x^x}{e^x}$$

$$y = \frac{1}{\frac{x^x}{e^x}} \int \frac{x^x}{e^x} \cdot x^{-x} \, dx$$

$$= \frac{e^x}{x^x} \int e^{-x} \, dx$$

$$= \frac{e^x}{x^x} \left[-e^{-x} + C \right]$$

$$y = -\frac{1}{x^x} + C \frac{e^x}{x^x}$$

$$y = \frac{Ce^x - 1}{x^x}$$

USING (2, 0)

$$0 = \frac{Ce^2 - 1}{4}$$

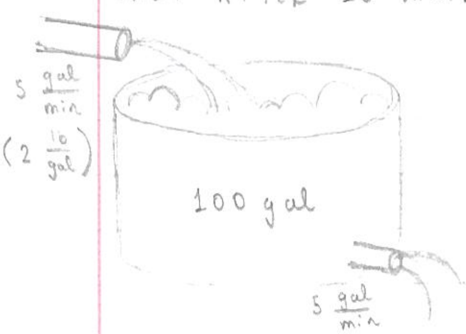
$$0 = Ce^2 - 1$$

$$1 = Ce^2$$

$$C = \frac{1}{e^2}$$

$$y = \frac{\frac{e^x}{e^2} - 1}{x^x} = \boxed{\frac{e^{x-2} - 1}{x^x}}$$

EX 5) A 100 gal TANK OF WATER CONTAINS 4 lbs OF SALT. SUPPOSE $2 \frac{\text{lbs}}{\text{gal}}$ OF BRINE SOLUTION IS PUMPED INTO THE TANK AT A RATE OF $5 \frac{\text{gal}}{\text{min}}$. THE SALT SOLUTION IS ALSO BEING DRAINED AT $5 \frac{\text{gal}}{\text{min}}$. FIND THE AMOUNT OF SALT AFTER 10 min.



LET $y(t)$ REPRESENT THE AMOUNT OF SALT IN THE 100 gal AT ANY TIME t.

WE ARE GIVEN $y(0) = 4$.

WE WANT $y(10)$

IN THIS PROBLEM, THE CHANGING AMOUNT OF SALT IS BASED ON THE AMOUNT OF SALT & THE AMOUNT OF SALT GOING OUT.

$$\therefore \frac{dy}{dt} = \text{RATE IN} - \text{RATE OUT}$$

$$\text{RATE}_{\substack{\text{IN}\\\text{SALT}}} = \left[\frac{5\ \cancel{gal}}{1\text{min}}\right] \cdot \left[\frac{2\text{ lb}}{1\ \cancel{gal}}\right] = 10\ \frac{\text{lb}}{\text{min}}$$

$$\text{RATE}_{\substack{\text{OUT}\\\text{SALT}}} = \left[\frac{5\ \cancel{gal}}{1\text{ min}}\right] \cdot \left[\frac{y(t)\text{ lb}}{100\ \cancel{gal}}\right] = \frac{y(t)}{20}\ \frac{\text{lb}}{\text{min}}$$

SINCE $\frac{dy}{dt} = \text{RATE}_{\text{IN SALT}} - \text{RATE}_{\text{OUT SALT}}$

$$\frac{dy}{dt} = 10 - \frac{y}{20}$$

$$\frac{dy}{dt} + \frac{1}{20}y = 10$$

$p(t) = \frac{1}{20}$

$q(t) = 10$

$$\mu = e^{\int \frac{1}{20}dt} = e^{\frac{t}{20}}$$

$$y = \frac{1}{e^{\frac{t}{20}}}\int e^{\frac{t}{20}} \cdot 10\,dt$$

$$= e^{-\frac{t}{20}}\left[e^{\frac{t}{20}} \cdot 10 \cdot 20\right]$$

$$y = e^{-\frac{t}{20}}\left[200e^{\frac{t}{20}} + C\right]$$

$$y(t) = 200 + Ce^{-\frac{t}{20}}$$

USING $(0, 4)$

$4 = 200 + Ce^{0}$

$4 = 200 + C$

$C = -196$

$$y(t) = 200 - 196e^{-\frac{t}{20}}$$

$$y(10) = 200 - 196e^{-\frac{10}{20}}$$

$$= 200 - 196e^{-\frac{1}{2}}$$

$$\approx \boxed{81.1\text{ lb}}$$