# Derivatives and Integrals Involving Logarithmic Functions
# Solutions To Selected Problems
# Calculus $9^{th}$ Edition Anton, Bivens, Davis

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January 20, 2012

1. $$y = \ln 5x$$

$$\frac{dy}{dx} = \frac{1}{5x} \cdot 5 = \boxed{\frac{1}{x}}$$

3. $$y = \ln|1 + x|$$

$$y = \ln(1 + x) \quad For\ x > 0$$
$$\frac{dy}{dx} = \frac{1}{1 + x} \cdot 1 = \boxed{\frac{1}{1 + x}}$$

5. $$y = \ln|x^2 - 1| = \ln(x^2 - 1), \quad For\ all\ x.$$

$$\frac{dy}{dx} = \frac{1}{x^2 - 1} \cdot (2x) = \boxed{\frac{2x}{x^2 - 1}}$$

7. $$y = \ln\left(\frac{x}{1 + x^2}\right) = \ln(x) - \ln(1 + x^2)$$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{1 + x^2} \cdot 2x$$

$$= \frac{x^2 + 1 - 2x^2}{x(1 + x^2)} = \boxed{\frac{1 - x^2}{x(1 + x^2)}}$$

11. $$y = \sqrt{\ln(x)} = (\ln(x))^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(\ln(x))^{-1/2} \cdot \frac{1}{x} = \boxed{\frac{1}{2x\sqrt{\ln(x)}}}$$

17. $$y = \frac{x^2}{1 + \log(x)}$$

$$\frac{dy}{dx} = \frac{2x(1 + \log(x)) - x^2\left(\frac{1}{x \ln 10}\right)}{(1 + \log(x))^2} = \boxed{\frac{2x(1 + \log(x)) - x \ln 10}{(1 + \log(x))^2}}$$

19. $$y = \ln(\ln(x))$$

$$\frac{dy}{dx} = \frac{1}{\ln(x)} \cdot \frac{1}{x} = \boxed{\frac{1}{x \ln(x)}}$$

23. $$y = \cos(\ln(x))$$

$$\frac{dy}{dx} = -\sin(\ln(x)) \cdot \frac{1}{x} = \boxed{-\frac{\sin(\ln(x))}{x}}$$

25. $$y = \log(\sin^2(x))$$

$$\frac{dy}{dx} = \frac{1}{\sin^2(x) \ln 10} \cdot (2\sin(x)) \cdot \cos(x)$$

$$= \frac{2\cos(x)}{\sin(x) \ln 10} = \boxed{\frac{2\cot(x)}{\ln 10}}$$

27. $$\frac{d}{dx}\left[\ln((x-1)^3(x^2+1)^4)\right]$$

$$= \frac{d}{dx}\left[\ln(x-1)^3 + \ln(x^2+1)^4\right]$$

$$= \frac{d}{dx}\left[3\ln(x-1) + 4\ln(x^2+1)\right]$$

$$= \frac{3}{x-1} + \frac{4}{x^2+1}\cdot 2x = \boxed{\frac{3}{x-1} + \frac{8x}{x^2+1}}$$

37. Find $dy/dx$ using logarithmic differentiation.

$$y = \frac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}$$

$$\ln(y) = \ln\left(\frac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}\right)$$

$$= \frac{1}{3}\ln(x^2-8) + \frac{1}{2}\ln(x^3+1) - \ln(x^6-7x+5)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(\frac{1}{3}\ln(x^2-8) + \frac{1}{2}\ln(x^3+1) - \ln(x^6-7x+5)\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2x}{3(x^2-8)} + \frac{3x^2}{2(x^3+1)} - \frac{6x^5-7}{x^6-7x+5}$$

$$\frac{dy}{dx} = y\cdot\left(\frac{2x}{3(x^2-8)} + \frac{3x^2}{2(x^3+1)} - \frac{6x^5-7}{x^6-7x+5}\right)$$

$$\boxed{\frac{dy}{dx} = \left[\frac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}\right]\cdot\left[\frac{2x}{3(x^2-8)} + \frac{3x^2}{2(x^3+1)} - \frac{6x^5-7}{x^6-7x+5}\right]}$$

41. Find the equation of the tangent line to the graph of $f(x) = \ln(x)$ at $x_0 = e^{-1}$

$$f(x) = \ln(x)$$
$$f'(x) = \frac{1}{x}$$

$$f'(e^{-1}) = \frac{1}{e^{-1}} = e$$

Now using the point slope formula:

$$y - (-1) = e(x - e^{-1})$$
$$y + 1 = ex - 1$$
$$\boxed{y = ex - 2}$$

47. Find a formula for the area $A(w)$ of the triangle bounded by the tangent line to the graph of $y = \ln(x)$ at $P(w, \ln(w))$, the horizontal line through $P$, and the $y$-axis.

First find the equation of the tangent line to $y = \ln(x)$ at $x = w$. Then find the $b$-intercept, which occurs when $x = 0$.

$$\left.\frac{dy}{dx}\right|_{x=w} = \frac{1}{w}$$

$$y - \ln(w) = \frac{1}{w}(x - w)$$

$$y = \frac{x}{w} - 1 + \ln(w)$$

$$b_{int} = \frac{0}{w} - 1 + \ln(w) = \ln(w) - 1$$

The width of the triangle is $w$ and the height is given by
$h = \ln(w) - (\ln(w) - 1) = 1$. So the area will be $\boxed{A(w) = \frac{1}{2} \cdot w \cdot 1 = \frac{w}{2}}$

53. (a) Find the limit by interpreting the expression as an appropriate derivative.

$$\lim_{x \to 0} \frac{\ln(1+3x)}{x}$$

Let $f(x) = \ln(1+3x)$. Note that $f(0) = \ln(1) = 0$. Then we can rewrite the limit as follows:

$$\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(x+h) - f(0)}{h} = f'(0)$$

$$= \frac{d}{dx} f(x)\bigg|_{x=0} = \frac{3}{1+3x}\bigg|_{x=0} = \frac{3}{1+0} = \boxed{3}$$

55. (b) Find the limit by interpreting the expression as an appropriate derivative.

$$\lim_{h \to 0} \frac{(1+h)^{\sqrt{2}} - 1}{h}$$

Let $f(x) = x^{\sqrt{2}}$, then $f(1) = 1$. Then we get :

$$\lim_{h \to 0} \frac{(1+h)^{\sqrt{2}} - 1}{h} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = f'(1)$$

$$= \frac{d}{dx} f(x)\bigg|_{x=1} = \sqrt{2}x^{\sqrt{2}-1}\bigg|_{x=1} = \boxed{2}$$

63. $$\int \frac{t+1}{t}\,dt = \int \frac{t}{t} + \frac{1}{t}\,dt = \int 1 + \frac{1}{t}\,dt = \boxed{t + \ln|t| + c}$$

65. $$\int_0^2 \frac{3x}{1+x^2}\,dx \quad u = 1 + x^2, \quad x = 0 \to u = 1, \quad x = 2 \to u = 5$$

$$du = 2x\,dx$$

$$\frac{3}{2}\,du = 3x dx$$

$$\frac{3}{2}\int_1^5 \frac{1}{u}\,du = \left(\frac{3}{2}\ln(u)\Big|_1^5\right)$$

$$= \frac{3}{2}\left(\ln(5) - \ln(1)\right) = \boxed{\frac{3}{2}\ln(5)}$$