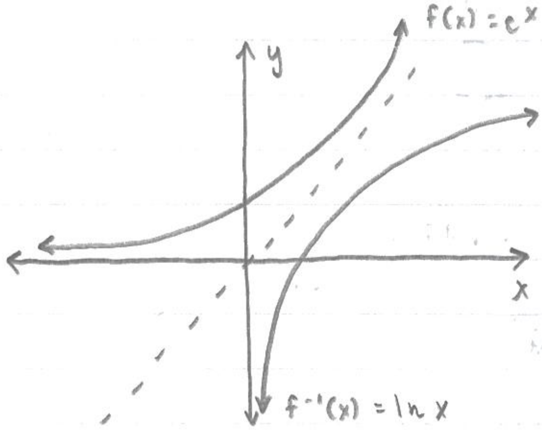
INVERSE FUNCTIONS AND THEIR DERIVATIVES

★ $\boxed{f^{-1}(x) \neq \frac{1}{f(x)}}$

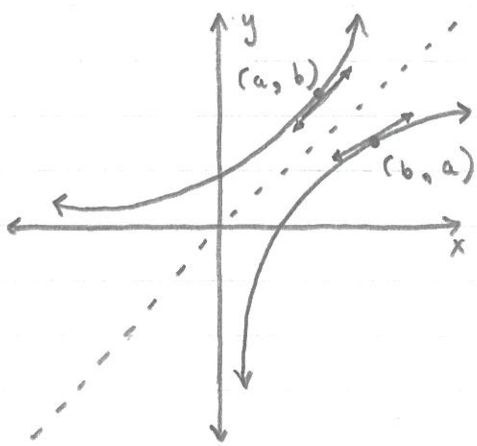


$f(x) = e^x$

$f^{-1}(x) = \ln x$

$f[f^{-1}(x)] = e^{\ln x} = x$

$f^{-1}[f(x)] = \ln e^x = x$



$f(a) = b$

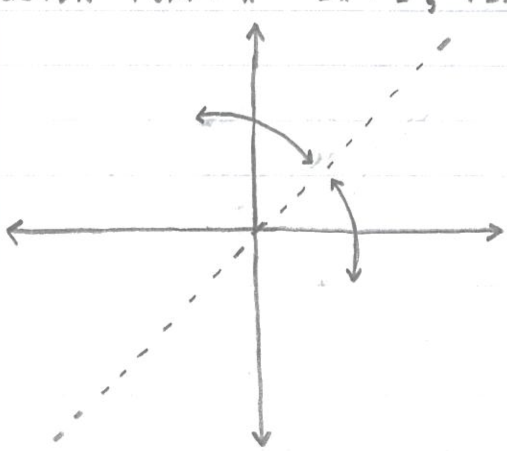
$f^{-1}(b) = a$

★ $\boxed{(f^{-1})'(b) = \frac{1}{f'(a)}}$

- OR -

$(f^{-1})'(b) = \frac{1}{f'[f^{-1}(b)]}$

EX1: GIVEN $f(x) = x^3 - 2x + 1$, FIND $(f^{-1})'(1)$.



$(f^{-1})'(1) = \frac{1}{f'[f^{-1}(b)]}$

$(f^{-1})'(1) = \frac{1}{f'(a)}$

WE KNOW $f'(x) = 3x^2 - 2$, BUT WE NEED THE VALUE OF a.

USING $(a, 1)$ ON $f(x)$, WE CAN FIND a.

$1 = a^3 - 2a + 1$

BY INSPECTION, $a = 0$

$\therefore (f^{-1})(1) = \dfrac{1}{f'(0)} = \dfrac{1}{3(0)^2 - 2} = -\dfrac{1}{2}$

HWK #1 LET $f(x) = x^5 + x^3 + 1$.

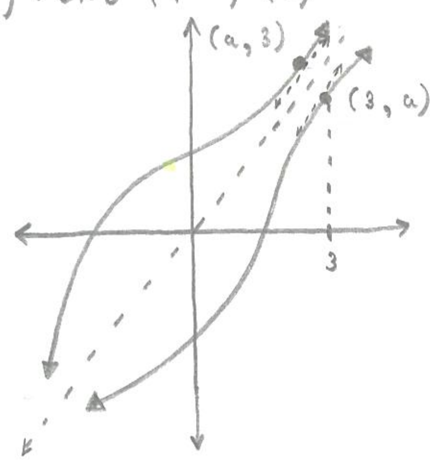
6.3 a) SHOW THAT $f(x)$ IS ONE-TO-ONE.

$f'(x) = 5x^4 + 3x^2 > 0$

$f(x)$ IS ALWAYS INCREASING,

$\therefore f(x)$ IS ONE-TO-ONE.

b) FIND $(f^{-1})'(3)$



$(f^{-1})'(3) = \dfrac{1}{f'(a)}$

$f'(x) = 5x^4 + 3x^2$

WE NEED TO FIND a.

USING $(a, 3)$ ON $f(x)$.

$3 = a^5 + a^3 + 1$

$a = 1$ BY INSPECTION.

$(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{5(1)^4 + 3(1)^2} = \dfrac{1}{8} \Rightarrow f'(3) = 8$

LOGARITHMIC DIFFERENTIATION

WHAT IS $\frac{d}{dx}[3^x]$?

$$\frac{d}{dx}[3^x] = \boxed{3^x \ln 3}$$

USING LOGARITHMIC DIFFERENTIATION...

LET $y = 3^x$

$\ln y = \ln 3^x$

$\ln y = x \ln 3$

$\frac{d}{dx}[\ln y] = \frac{d}{dx}[x \ln 3]$

$\frac{1}{y} \cdot \frac{dy}{dx} = \ln 3$

$\frac{dy}{dx} = y \ln 3$

$\frac{dy}{dx} = \boxed{3^x \ln 3}$

NOTE: $3^x = e^{\ln 3^x} = e^{x \ln 3}$

$\frac{d}{dx}[3^x] = \frac{d}{dx}[e^{x\ln 3}]$

$= e^{x \ln 3} \cdot \ln 3$

$= \boxed{3^x \ln 3}$

FIND ANY HORIZANTAL TANGENT LINES OF $f(x) = x^x$, $x > 0$.

Let $y = x^x$

$\ln y = \ln x^x$

$\ln y = x \ln x$

$\frac{d}{dx}[\ln y] = \frac{d}{dx}[x \ln x]$

$\frac{1}{y} \cdot \frac{dy}{dx} = \ln x + x \cdot \frac{1}{x}$

$= y[\ln x + 1]$

$= \boxed{x^x[\ln x + 1]}$

NOTE: $\frac{d}{dx}[x^x] = \frac{d}{dx}[e^{x \ln x}]$

$= e^{x \ln x}\left[1 \cdot \ln x + x \cdot \frac{1}{x}\right]$

$= \boxed{x^x[\ln x + 1]}$

NOW WE SET $f'(x) = 0$

$\ln x + 1 = 0$

$\ln x = -1$

$e^{-1} = x$

$\boxed{x = \frac{1}{e}}$

#35) DIFFERENTIATE.

$y = (\ln x)^{\ln x}$

$\ln y = \ln (\ln x)^{\ln x}$

$\ln y = \underbrace{\ln x}\cdot\underbrace{\ln(\ln x)}$

$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x}[\ln(\ln x)] + \cancel{\ln x}^{1}\left[\frac{1}{\cancel{\ln x}}\cdot\frac{1}{x}\right]$

$\frac{dy}{dx} = y\left[\frac{\ln(\ln x)}{x} + \frac{1}{x}\right]$

$\boxed{\frac{dy}{dx} = (\ln x)^{\ln x}\cdot\left[\frac{\ln(\ln x) + 1}{x}\right]}$

WHEN DOES $\frac{dy}{dx} = 0$?

$\ln(\ln x) + 1 = 0$

$\ln(\ln x) = -1$

$\ln x = e^{-1}$

$\ln x = \frac{1}{e}$

$\boxed{x = e^{\frac{1}{e}}}$

#29) DIFFERENTIATE.

$$y = \pi^{\sin x}$$
$$\ln y = \ln \pi^{\sin x}$$
$$\ln y = \sin x \cdot \ln \pi$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \cdot \ln \pi$$
$$\frac{dy}{dx} = y\left[\cos x \cdot \ln \pi\right]$$
$$\boxed{\frac{dy}{dx} = \pi^{\sin x}\left[\cos x \cdot \ln \pi\right]}$$

- OR -

$$y = e^{\sin x \ln \pi}$$
$$\boxed{\frac{dy}{dx} = \pi^{\sin x}\left[\cos x \cdot \ln \pi\right]}$$

MORE EXPLICITLY...

$$y = e^{\sin x \ln \pi}$$
$$\frac{dy}{dx} = e^{\sin x \ln \pi}\left[\cos x \ln \pi\right]$$
$$= \cancel{e^{\ln \pi^{\sin x}}}\left[\cos x \ln \pi\right] = \boxed{\pi^{\sin x}\left[\cos x \ln \pi\right]}$$

#69) $\displaystyle\int x^2 e^{-2x^3}\, dx$

let $u = -2x^3$

$du = -6x^2\, dx$

$-\frac{1}{6}\, du = x^2\, dx$

$$-\frac{1}{6}\int e^u\, du = -\frac{1}{6} e^u + C = \boxed{-\frac{e^{-2x^3}}{6} + C}$$

#79) $\displaystyle\int_{-\ln 3}^{\ln 3} \left[\frac{e^x}{e^x + 4}\right] dx$

let $u = e^x + 4$

$du = e^x\, dx$

$x = \ln 3 \Rightarrow u = e^{\ln 3} + 4 = 7$

$x = -\ln 3 \Rightarrow u = e^{-\ln 3} + 4 = \frac{13}{3}$

$$\int_{\frac{13}{3}}^{7} \frac{1}{u}\, du = \ln u \Big|_{\frac{13}{3}}^{7}$$
$$= \ln 7 - \ln \frac{13}{3}$$
$$= \boxed{\ln \frac{21}{13}}$$

SOLVE THE FOLLOWING EQUATION FOR X.

$$2\left[\frac{e^x - e^{-x}}{2}\right] = [1]\,2$$

$$e^x - e^{-x} = 2$$

$$e^x\left[e^x - \frac{1}{e^x}\right] = [2]\,e^x$$

$$e^{2x} - 1 = 2e^x$$

$$e^{2x} - 2e^x - 1 = 0$$

let $u = e^x$

$$u^2 - 2u - 1 = 0$$

$$u^2 - 2u = 1$$

$$u^2 - 2u + 1 = 1 + 1$$

$$u^2 - 2u + 1 = 2$$

$$(u - 1)^2 = 2$$

$$u - 1 = \pm\sqrt{2}$$

$$u = 1 \pm \sqrt{2}$$

$$\downarrow$$

$$e^x = 1 + \sqrt{2}$$

$$\boxed{x = \ln(1 + \sqrt{2})}$$