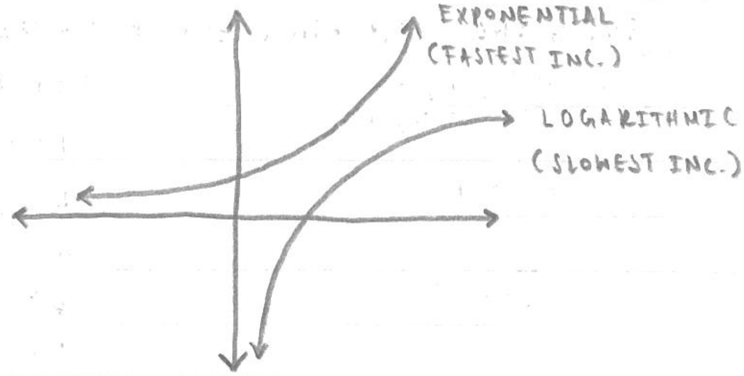
6.4 GRAPHS AND APPLICATIONS INVOLVING LOG AND EXPONENTIAL FUNCTIONS

NOTE: $\lim_{x \to +\infty} \left[\frac{e^x}{x^n} \right] = +\infty$



$\lim_{x \to +\infty} \left[\frac{x^n}{e^x} \right] = 0$

$\lim_{x \to +\infty} \left[\frac{\ln 2x}{x^{100}} \right] = 0$

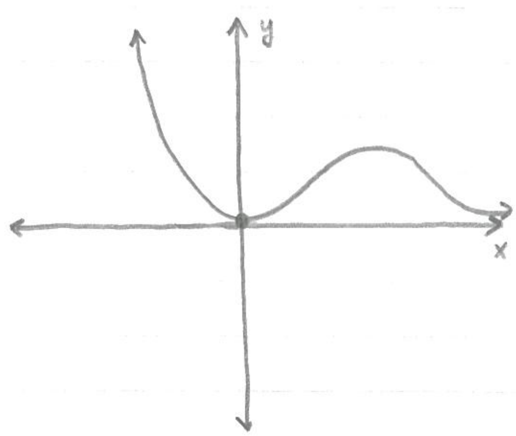
* $e^x > x > \log x$

$\lim_{x \to -\infty} \left[x e^x \right] = 0$

$\lim_{x \to 0^+} \left[x^{100} \cdot \ln x \right] = 0$

#11) GRAPH $f(x) = x^2 \cdot e^{-2x}$

$\lim_{x \to +\infty} \left[x^2 e^{-2x} \right] = \lim_{x \to +\infty} \left[\frac{x^2}{e^{2x}} \right] = 0$



$\lim_{x \to -\infty} \left[x^2 e^{-2x} \right] = +\infty$

SIGN CHART FOR $f(x)$

x^2 + + + + 0 + + + +

e^{-2x} + + + + + + + + + ← ALWAYS POS.

(+) 0 (+)

$f'(x) = 2x e^{-2x} + x^2 \cdot e^{-2x} \cdot (-2)$

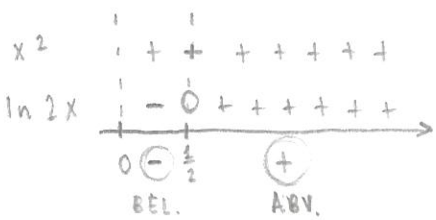
$= 2xe^{-2x} - 2x^2 e^{-2x}$

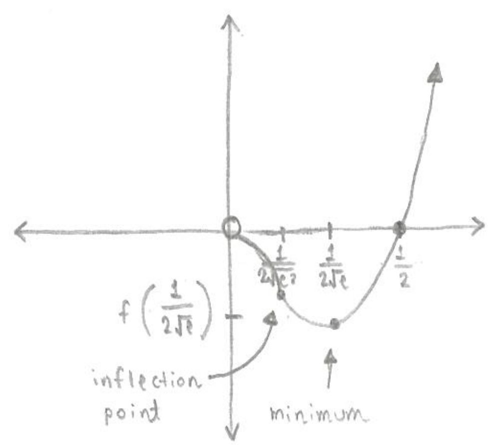
$= 2xe^{-2x}(1 - x)$

$f'(x) = 0$ WHEN $x = 0, 1$

$$f''(x) = [2e^{-2x} + 2xe^{-2x} \cdot (-2)](1-x) + 2xe^{-2x}(-1)$$
$$= [2e^{-2x} - 4xe^{-2x}](1-x) - 2xe^{-2x}$$
$$= 2e^{-2x}[1 - 2x](1-x) - x$$
$$= 2e^{-2x}[1 - 3x - 2x^2 - x]$$
$$= 2e^{-2x}(\underbrace{2x^2 - 4x + 1})$$
$$\downarrow$$
$$2x^2 - 4x + 1 = 0$$
$$x = \frac{-(-4) \pm \sqrt{16 - (4)(2)(1)}}{2(2)}$$
$$x = \frac{4 \pm \sqrt{8}}{4} = \frac{4 \pm 2\sqrt{2}}{4} = \frac{2 \pm \sqrt{2}}{2} = \boxed{1 \pm \frac{\sqrt{2}}{2}}$$

#21. GRAPH $f(x) = x^2 \ln 2x$, $x > 0$





$\ln 2x > 0$ e
$2x > 1$
$x > \frac{1}{2}$

END BEHAVIOR

$\lim_{x \to 0^+} x^2 \ln(2x) = 0$

GIVEN

$\lim_{x \to +\infty} x^2 \ln(2x) = +\infty$

INCREASING: $\left(\frac{1}{2\sqrt{e}}, +\infty\right)$
DECREASING: $\left(0, \frac{1}{2\sqrt{e^3}}\right)$
CONCAVE UP: $\left(\frac{1}{2\sqrt{e^3}}, +\infty\right)$
CONCAVE DWN: $\left(0, \frac{1}{2\sqrt{e^3}}\right)$

DOMAIN: $(0, +\infty)$
RANGE: $\left(f\left(\frac{1}{2\sqrt{e}}\right), +\infty\right)$

$$f'(x) = 2x\ln 2x + x^2 \cdot \frac{1}{2x}\cdot 2 \to \frac{1}{x} \to x$$
$$= 2x\ln(2x) + x$$
$$= x(2\ln 2x + 1)$$

$$f'(x) = 2x\ln(2x) + x$$
$$f''(x) = 2\ln(2x) + \cancel{2x}\cdot\frac{1}{x} + 1 \to 2$$
$$= 2\ln(2x) + 3$$

x: + + + + + + + +
$2\ln 2x + 1$: − − 0 + + + + +
0 ⊖ $\frac{1}{2\sqrt{e}}$ ⊕
DEC. ↑ MIN. INC.

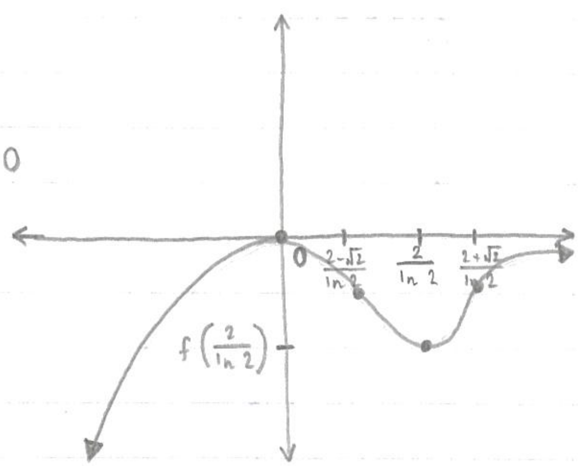
$2\ln(2x) + 3$: − − − + + + + + +
0 ⊖ $\frac{1}{\sqrt{e^3}}$ ⊕
CD ↑ Inf. pt. CU

$$2\ln 2x + 1 \geq 0$$
$$2\ln 2x \geq -1$$
$$e^{\ln(2x)} \geq e^{-\frac{1}{2}}$$
$$2x \geq \frac{1}{\sqrt{e}}$$
$$x = \frac{1}{2\sqrt{e}}$$

$$2\ln(2x) + 3 \geq 0$$
$$2\ln(2x) \geq -3$$
$$e^{\ln(2x)} \geq e^{-\frac{3}{2}}$$
$$2x \geq \frac{1}{(\sqrt{e})^3}$$
$$x \geq \frac{1}{2\sqrt{e^3}}$$

* EX#2) GRAPH $f(x) = -x^2 \cdot 2^{-x}$

$-x^2$: − − − − 0 − − − − −
2^{-x}: + + + + + + + + + ← NEVER EQUALS 0
⊖
ALWAYS BELOW



$$f'(x) = -2x\cdot 2^{-x} + (-x^2)(2^{-x}\ln 2)(-1)$$
$$= -2x\,2^{-x} + x^2\,2^{-x}\ln 2$$
$$= -x\cdot 2^{-x}(2 - x\ln 2)$$

$-x$: + + + + + + 0 − − − − −
$2 - x\ln 2$: + + + + + + + + 0 − − −
⊕ 0 ⊖ $\frac{2}{\ln 2}$ ⊕
INC. ↑ DEC. ↑ INC.
MAX MIN

$$2 - x\ln 2 \geq 0$$
$$-x\ln 2 \geq -2$$
$$x\ln 2 \leq 2$$
$$x \leq \frac{2}{\ln 2}$$

D: $\mathbb{R}$
R: $(-\infty, 0]$
INC: $(-\infty, 0) \cup \left(\frac{2}{\ln 2}, +\infty\right)$
DEC: $\left(0, \frac{2+\sqrt{2}}{\ln 2}\right)$
CU: $\left(\frac{2-\sqrt{2}}{\ln 2}, \frac{2+\sqrt{2}}{\ln 2}\right)$
CD: $\left(-\infty, \frac{2-\sqrt{2}}{\ln 2}\right) \cup \left(\frac{2+\sqrt{2}}{\ln 2}, +\infty\right)$

$$f'(x) = \underbrace{-x \cdot 2^{-x}}\,\underbrace{[2 - x\ln 2]}$$

$$f''(x) = \{(-1)(2^{-x}) + (-x)(2^{-x})(\ln 2)(-1)\}[2 - x\ln 2] + \underbrace{(-x)2^{-x}(-\ln 2)}$$

$$= [-2^{-x} + x2^{-x} \cdot \ln 2][2 - x\ln 2] + x\ln 2 \cdot 2^{-x}$$

$$= -2^{-x}\{[1 - x\ln 2][2 - x\ln 2] - x\ln 2\}$$

$$= -2^{-x}\{2 - x\ln 2 - 2x\ln 2 + (x\ln 2)^2 - x\ln 2\}$$

$$= -2^{-x}\{(x\ln 2)^2 - 4x\ln 2 + 2\}$$

$$(x\ln 2)^2 - 4x\ln 2 + 2 = 0$$

$u = x\ln 2$

$u^2 - 4u + 2 = 0$

$u^2 - 4u + 4 = -2 + 4$

$(u - 2)^2 = 2$

$u - 2 = \pm\sqrt{2}$

$u = 2 \pm \sqrt{2}$

$2 \pm \sqrt{2} = x\ln 2$

$$x = \frac{2 \pm \sqrt{2}}{\ln 2}$$

-2^{-x}	$- - -$		$- - - - - -$		$- - - - -$
$(x\ln 2)^2 - 4x\ln 2 + 2$	$+ + +$	0	$- - - - - -$	0	$+ + + +$ ← PARABOLIC BEHAVIOR
	$-$	$\frac{2-\sqrt{2}}{\ln 2}$	$+$	$\frac{2+\sqrt{2}}{\ln 2}$	$-$
	DEC	↑ I.P.	INC	↑ I.P	DEC.