6.5 L'HÔPITAL'S RULE; INDETERMINATE FORMS 06/05

RECALL: $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$

→ THIS LIMIT IS OF THE INDETERMINANT FORM $1^{+\infty}$

- INDETERMINANT FORMS OF LIMITS ARE:

$$\frac{\infty}{\infty},\ \frac{0}{0},\ 1^{\pm\infty},\ \infty - \infty,\ 0^0,\ \infty^0,\ 0 \cdot \infty$$

- L'HÔPITAL'S RULE:

→ If you have a limit in the indeterminate form $\frac{\infty}{\infty}$ or $\frac{0}{0}$, then you can apply L'Hôpital's Rule which is:

$$\star\ \boxed{\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

SUPPOSE WE HAVE:

a) $\lim\limits_{x \to +\infty} \left(\frac{2^x}{x}\right) \overset{L'H}{=} \lim\limits_{x \to +\infty} \left[\frac{2^x \cdot \ln 2}{1}\right] = +\infty$

b) $\lim\limits_{x \to 0} \left[\frac{\sin x}{x}\right] \overset{L'H}{=} \lim\limits_{x \to 0} \left[\frac{\cos x}{1}\right] = 1$

c) $\lim\limits_{x \to 0} \left[\frac{\tan 7x}{8x}\right] \overset{L'H}{=} \lim\limits_{x \to 0} \left[\frac{\sec^2 7x \cdot 7}{8}\right] = \frac{7}{8}\left[\lim\limits_{x \to 0} \sec 7x\right]^2 = \frac{7}{8}(1) = \frac{7}{8}$

EX #1) a) $\lim\limits_{x \to +\infty} \frac{e^x}{x^2} \overset{L'H}{=} \lim\limits_{x \to +\infty} \left[\frac{e^x}{2x}\right] \overset{L'H}{=} \lim\limits_{x \to +\infty} \left[\frac{e^x}{2}\right] = \frac{+\infty}{2} = +\infty$

b) $\lim\limits_{x \to 0^+} (x^2 \ln x) = \lim\limits_{x \to 0^+} \left[\frac{\ln x}{\frac{1}{x^2}}\right] \overset{L'H}{=} \lim\limits_{x \to 0^+} \left[\frac{\frac{1}{x}}{-\frac{2}{x^3}}\right] = \lim\limits_{x \to 0^+} \left[-\frac{x^2}{2}\right] = 0$

↑ $0 * -\infty$

↑ $\frac{-\infty}{+\infty}$

c) $\lim\limits_{x \to 0^+} [x^x]$

$= \lim\limits_{x \to 0^+} e^{x \ln x}$

$= e^{\lim_{x \to 0^+} (x \ln x)}$

$= e^{\lim_{x \to 0^+} \left[\frac{\ln x}{\frac{1}{x}} \right]}$

$\overset{L'H}{=} e^{\lim_{x \to 0^+} \left[\frac{\frac{1}{x}}{-\frac{1}{x^2}} \right]}$

$= e^{\lim_{x \to 0^+} (-x)}$

$= e^0$

$= 1$

- OR -

SAME LIMIT USING THE NATURAL LOG FUNCTION:

let $y = x^x$

$\ln y = \ln x^x$

$\lim\limits_{x \to 0^+} (\ln y) = \lim\limits_{x \to 0^+} (\ln x^x)$

$= \lim\limits_{x \to 0^+} (x \ln x)$

$= \lim\limits_{x \to 0^+} \left[\frac{\ln x}{\frac{1}{x}} \right]$

$\overset{L'H}{=} \lim\limits_{x \to 0^+} \left[\frac{\frac{1}{x}}{-\frac{1}{x^2}} \right]$

$= \lim\limits_{x \to 0^+} [-x]$

$= 0$

$\therefore e^{\lim_{x \to 0^+} (\ln y)} = e^0$

$\lim\limits_{x \to 0^+} y = e^0$

$\boxed{\lim\limits_{y \to 0^+} x^x = 1}$

EX#2) EVALUATE:

$$\lim_{x\to+\infty}(1+x)^{\frac{1}{x}} \to \infty^0$$

LET $y=(1+x)^{\frac{1}{x}}$

$$\ln y = \tfrac{1}{x}\ln(1+x)$$

$$\lim_{x\to+\infty}(\ln y) = \lim_{x\to+\infty}\left[\frac{\ln(1+x)}{x}\right] \to \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to+\infty}\left[\frac{\frac{1}{1+x}}{1}\right]$$

$$= \lim_{x\to+\infty}\left(\frac{1}{1+x}\right)$$

$$= 0$$

$$e^{\lim_{x\to+\infty}(\ln y)} = e^0$$

$$\boxed{\lim_{x\to+\infty} y = 1}$$

RECALL: $\lim_{x\to+\infty}\left[1+\frac{1}{x}\right]^x \quad 1^\infty$

$$y=\left[1+\frac{1}{x}\right]^x$$

$$\ln y = x\ln\left(1+\tfrac{1}{x}\right)$$

$$\lim_{x\to+\infty}(\ln y) = \lim_{x\to+\infty}\left[\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right] \to \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to+\infty}\left[\frac{\left(\frac{1}{1+\frac{1}{x}}\right)\cdot\cancel{\left(-\frac{1}{x^2}\right)}^{1}}{\cancel{\left(-\frac{1}{x^2}\right)}}\right]$$

$$= \lim_{x\to+\infty}\left[\frac{1}{1+\frac{1}{x}}\right]$$

$$= 1$$

$$e^{\lim_{x\to+\infty}(\ln y)} = e^1$$

$$\boxed{\lim_{x\to+\infty} y = e}$$

EX#3) $\lim_{x \to \pi} \left[\frac{\sin \pi}{x - \pi} \right] \to \frac{0}{0}$

$\overset{L'H}{=} \lim_{x \to \pi} \left[\frac{\cos \overset{-1}{\pi}}{1} \right]$

$= -1$

EX#4) $\lim_{x \to 0} \left[\frac{\sin 2x}{\sin 5x} \right] \to \frac{0}{0}$

$\overset{L'H}{=} \lim_{x \to 0} \left[\frac{2 \cos 2x}{5 \cos 5x} \right]$

$= \frac{2}{5} \lim_{x \to 0} \left[\frac{\cos 2x}{\cos 5x} \right]$

$= \frac{2}{5}$

EX#5) $\lim_{x \to 0^+} [\tan x - \ln x] = 0 - (-\infty) = +\infty$

EX#6) $\lim_{x \to 0^+} [\tan x \cdot \ln x] \to 0 \cdot (-\infty)$

$= \lim_{x \to 0^+} \left[\frac{\ln x}{\cot x} \right] \to \frac{-\infty}{+\infty}$

$\overset{L'H}{=} \lim_{x \to 0^+} \left[\frac{\frac{1}{x}}{-\csc^2 x} \right] \to \frac{+\infty}{-\infty}$

$= \lim_{x \to 0^+} \left[-\frac{\sin^2 x}{x} \right]$

$= -\lim_{x \to 0^+} \left[\frac{\sin x}{x} \right] \cdot \lim_{x \to 0^+} (\sin x)$

$= -(1)(0)$

$= \boxed{0}$

EX#7) $\lim_{x \to +\infty} \left(1 - \frac{3}{x}\right)^x \to 1^\infty$

** NOTE: $\lim_{x \to +\infty} \left[1 + \frac{a}{x}\right]^x = e^a$

$\lim_{x \to +\infty} \left[1 + \frac{5}{x}\right]^x = e^5$

$\lim_{x \to +\infty} \left[1 - \frac{4}{x}\right]^x = e^{-4}$

LET $y = \left(1 - \frac{3}{x}\right)^x$

$\ln y = x \ln\left(1 - \frac{3}{x}\right)$

$\lim_{x \to +\infty} (\ln y) = \lim_{x \to +\infty} \left[\frac{\ln\left(1 - \frac{3}{x}\right)}{\frac{1}{x}}\right] \to \frac{0}{0}$

$\overset{L'H}{=} \lim_{x \to +\infty} \left[\frac{\left(\frac{1}{1-\frac{3}{x}}\right)\left(\frac{3}{\cancel{x^2}}\right)}{-\frac{1}{\cancel{x^2}}}\right]$

$= \lim_{x \to +\infty} \left[-\frac{3}{1 - \cancel{\frac{3}{x}}^{\,0}}\right]$

$= -3$

$\therefore \lim_{x \to +\infty} e^{(\ln y)} = e^{-3}$

$$\boxed{\lim_{x \to +\infty} (y) = \frac{1}{e^3}}$$

EX#8) $\lim_{x \to +\infty} \left[\sqrt{x^2 + x} - x\right] \to \infty - \infty$

$= \lim_{x \to +\infty} \left[\frac{\sqrt{x^2+x} - x}{1}\right]\left[\frac{\sqrt{x^2+x} + x}{\sqrt{x^2+x} + x}\right]$

$= \lim_{x \to +\infty} \left[\frac{\cancel{x^2} + x - \cancel{x^2}}{\sqrt{x^2+x} + x}\right]$

$= \lim_{x \to +\infty} \left[\frac{x}{\sqrt{x^2+x} + x}\right] \cdot \frac{\frac{1}{|x|}}{\frac{1}{|x|}}$

$= \lim_{x \to +\infty} \left[\frac{x}{\sqrt{x^2+x} + x}\right] \cdot \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}}$ (argument is positive $\to +\infty$)

$= \lim_{x \to +\infty} \left[\frac{\frac{x}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{x}{x^2}} + \frac{x}{\sqrt{x^2}}}\right] = \lim_{x \to +\infty} \left[\frac{1}{\sqrt{1 + \frac{1}{x}} + 1}\right] = \frac{1}{2}$

EX#9) $\lim\limits_{x \to +\infty} \left[\dfrac{x+1}{x+2} \right]^x \longrightarrow 1^{\infty}$

$$y = \left[\frac{x+1}{x+2} \right]^x$$

$$\ln y = x \ln \left[\frac{x+1}{x+2} \right]$$

$$\lim_{x \to +\infty} (\ln y) = \lim_{x \to +\infty} \left[\frac{\ln \left[\frac{x+1}{x+2} \right]}{\frac{1}{x}} \right] \longrightarrow \frac{0}{0}$$

$$\stackrel{L'H}{=} \lim_{x \to +\infty} \left[\frac{\left(\frac{x+2}{x+1} \right) \left[\frac{(x+2) - (x+1)}{(x+2)^2} \right]}{-\frac{1}{x^2}} \right]$$

$$= \lim_{x \to +\infty} \left[\frac{\frac{\cancel{x} + 2 - \cancel{x} - 1}{(x+1)(x+2)}}{-\frac{1}{x^2}} \right]$$

$$= \lim_{x \to +\infty} \left[\frac{\frac{1}{(x+1)(x+2)}}{-\frac{1}{x^2}} \right]$$

$$= - \lim_{x \to +\infty} \left[\frac{x^2}{(x+1)(x+2)} \right]$$

$$= -1$$

$$\therefore e^{\lim\limits_{x \to +\infty} (\ln y)} = e^{-1}$$

$$\boxed{\lim_{x \to +\infty} y = \frac{1}{e}}$$