L'Hôpital's Rule; Indeterminate Forms
Solutions To Selected Problems
Calculus 9^{th} Edition Anton, Bivens, Davis

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I will denote the use of L'Hôpital's Rule by $\overset{LH}{=}$

7. $$\lim_{x\to 0} \frac{e^x - 1}{\sin(x)} \to \frac{e^0 - 1}{\sin(0)} = \frac{0}{0}$$

$$\overset{LH}{=} \lim_{x\to 0} \frac{e^x}{\cos(x)} \to \frac{e^0}{\cos(0)} = \frac{1}{1} = \boxed{1}$$

15. $$\lim_{x\to 0^+} \frac{\cot(x)}{\ln(x)} \to \frac{\infty}{-\infty}$$

$$\overset{LH}{=} \lim_{x\to 0^+} \frac{-\csc^2(x)}{\frac{1}{x}} = \lim_{x\to 0^+} -\frac{x}{\sin^2(x)} \to \frac{0}{0}$$

$$\overset{LH}{=} \lim_{x\to 0^+} -\frac{1}{2\sin(x)\cos(x)} \to -\frac{1}{0} \to \boxed{-\infty}$$

25. $\lim_{x \to \infty} \left(1 - \frac{3}{x}\right)^x \to 1^\infty$

$$Let\ y = \left(1 - \frac{3}{x}\right)^x$$

$$\ln(y) = x \ln\left(1 - \frac{3}{x}\right) = \frac{\ln\left(1 - \frac{3}{x}\right)}{\frac{1}{x}} \to \frac{0}{0}\ as\ x \to \infty$$

$$\lim_{x \to \infty} \ln(y) \stackrel{LH}{=} \lim_{x \to \infty} \frac{\frac{1}{1-\frac{3}{x}} \cdot \left(0 - \left(-\frac{3}{x^2}\right)\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{\frac{3}{x^2(1-\frac{3}{x})}}{-\frac{1}{x^2}} = \lim_{x \to \infty} \frac{-3}{1 - \frac{3}{x}} \to -3$$

$$\lim_{x \to \infty} e^{\ln(y)} = e^{-3} = \boxed{\frac{1}{e^3}}$$

33. $$\lim_{x\to+\infty}(\sqrt{x^2-x}-x)=\lim_{x\to+\infty}\frac{(\sqrt{x^2-x}-x)}{1}\cdot\frac{(\sqrt{x^2-x}+x)}{(\sqrt{x^2-x}+x)}$$

$$=\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+x}+x}\to\frac{\infty}{\infty}$$

$$\overset{LH}{=}\lim_{x\to+\infty}\frac{1}{\frac{1}{2}(x^2+x)^{-1/2}(2x+1)+1}$$

$$=\lim_{x\to+\infty}\frac{1}{\frac{2x+1}{2\sqrt{x^2+x}}+1}=\lim_{x\to+\infty}\frac{1}{\frac{2x+1}{2\sqrt{x^2+x}}+1}\cdot\frac{2\sqrt{x^2+x}}{2\sqrt{x^2+x}}$$

$$=\lim_{x\to+\infty}\frac{2\sqrt{x^2+x}}{2x+1+2\sqrt{x^2+x}}=\lim_{x\to+\infty}\frac{2\sqrt{x^2+x}}{2x+1+2\sqrt{x^2+x}}\cdot\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}}$$

$$=\lim_{x\to+\infty}\frac{2\sqrt{1+\frac{1}{x}}}{2+\frac{1}{x}+2\sqrt{1+\frac{1}{x}}}\to\frac{2\sqrt{1}}{2+2\sqrt{1}}=\frac{2}{4}=\boxed{\frac{1}{2}}$$

This problem can actually be solved without using L'Hôpitals Rule:

$$\lim_{x\to+\infty}(\sqrt{x^2-x}-x)=\lim_{x\to+\infty}\frac{(\sqrt{x^2-x}-x)}{1}\cdot\frac{(\sqrt{x^2-x}+x)}{(\sqrt{x^2-x}+x)}$$

$$=\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+x}+x}=\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+x}+x}\cdot\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}}$$

$$=\lim_{x\to+\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}\to\frac{1}{\sqrt{1}+1}=\boxed{\frac{1}{2}}\quad\ddot\smile$$