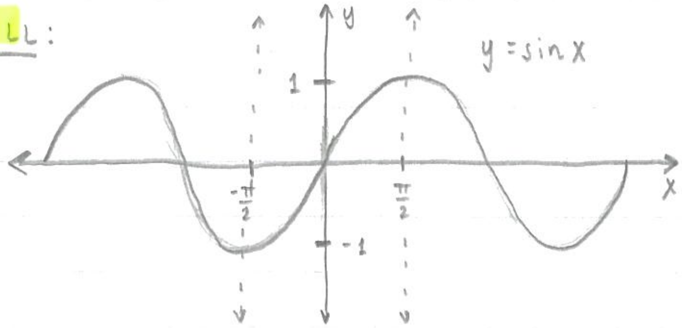
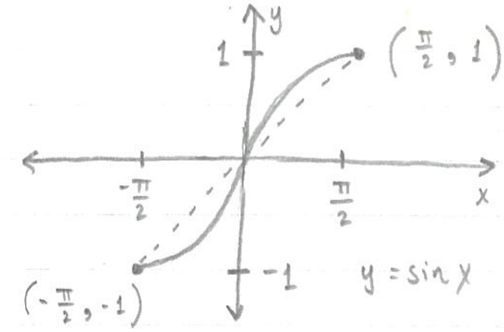
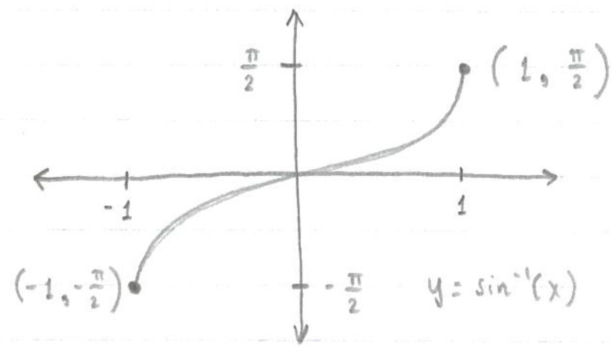
6.7 DERIVATIVES AND INTEGRALS INVOLVING INVERSE TRIG FUNCTIONS 06/05

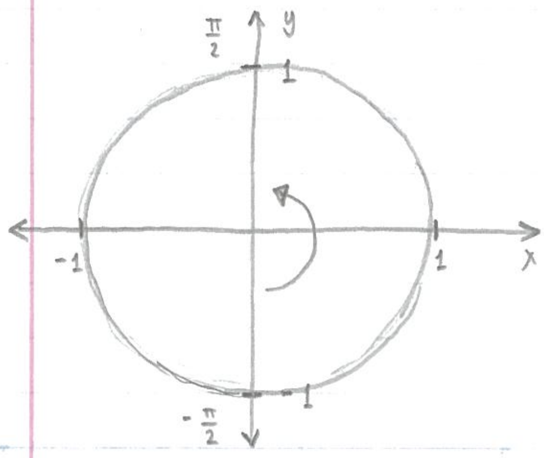
RECALL:





THIS FUNCTION IS NOT 1-1.
IF WE RESTRICT THE DOMAIN TO $[-\frac{\pi}{2}, \frac{\pi}{2}]$, THEN $y = \sin x$ IS 1-1.





$\sin \frac{11\pi}{6} = -\frac{1}{2}$ $\quad$ $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{2}$

$\sin \frac{7\pi}{6} = -\frac{1}{2}$ $\quad$ $\sin^{-1}(-1) = -\frac{\pi}{2}$

$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ $\quad$ $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

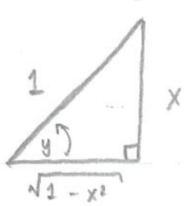
FIND $f'(x)$ GIVEN $f(x) = \sin^{-1}(x)$

let $y = \sin^{-1}(x) \longrightarrow \sin y = x$

$f'(x) = \frac{dy}{dx}$ $\qquad$ $\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$

$\cos y \cdot \frac{dy}{dx} = 1$

SINCE $x = \sin y$



$\frac{dy}{dx} = \frac{1}{\cos y}$

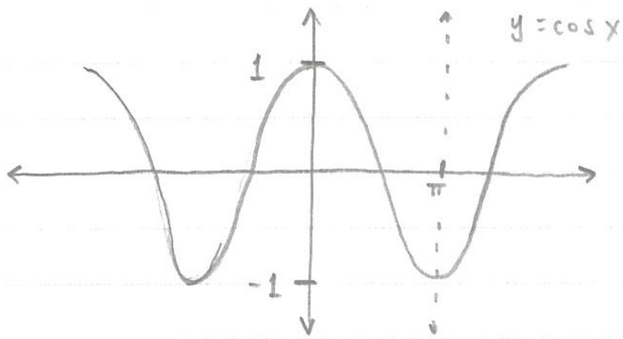
$= \boxed{\frac{1}{\sqrt{1-x^2}}}$

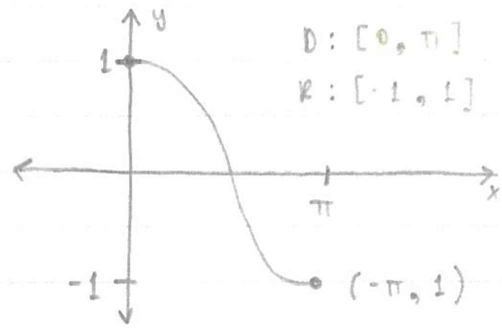
$* \quad \boxed{\therefore \frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}}$

AND

$* \quad \boxed{\therefore \int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C}$

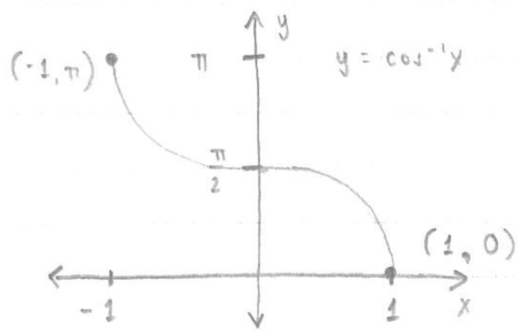
RECALL:





D: $[0, \pi]$
R: $[-1, 1]$

$y = \cos(x)$ IS 1-1 ON $[0, \pi]$



D: $[-1, 1]$
R: $[0, \pi]$

$$\cos^{-1}\left(-\tfrac{1}{2}\right) = \frac{2\pi}{3}$$

$$\cos^{-1}(-1) = \pi$$

$$\cos^{-1}\left(\tfrac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

FIND $f'(x)$ GIVEN $f(x) = \cos^{-1} x$

$y = \cos^{-1} x$

$x = \cos y$

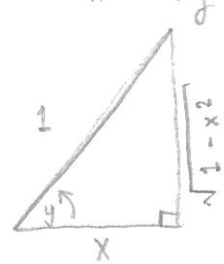
$$\frac{d}{dx}[x] = \frac{d}{dx}[\cos y]$$

$$1 = -\sin y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

$$= \boxed{-\frac{1}{\sqrt{1 - x^2}}}$$

$x = \cos y$

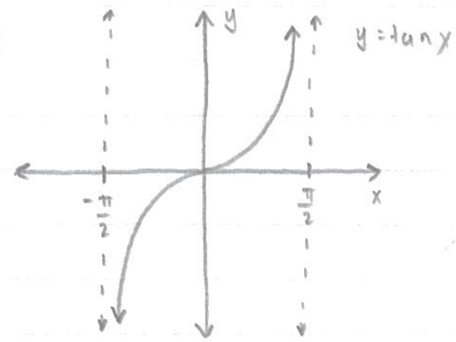


$$\therefore \frac{d}{dx}\left[\cos^{-1} x\right] = -\frac{1}{\sqrt{1 - x^2}}$$

AND

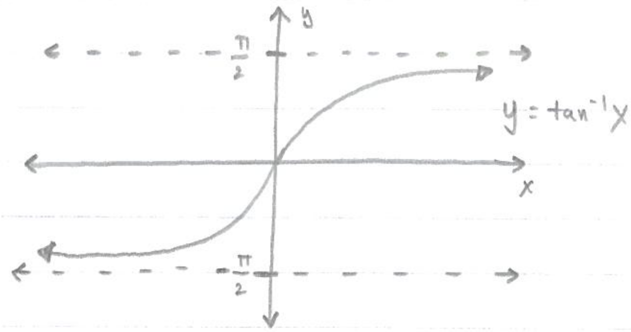
$$\therefore \int -\frac{1}{\sqrt{x^2 - 1}}\, dx = \cos^{-1} x + C = -\sin^{-1} x + C$$

RECALL:



D : $(-\frac{\pi}{2}, \frac{\pi}{2})$

R : $\mathbb{R}$



D : $\mathbb{R}$

R : $(-\frac{\pi}{2}, \frac{\pi}{2})$

* **NOTE**: $\lim\limits_{x \to +\infty} [\tan^{-1} x] = \frac{\pi}{2}$

$\lim\limits_{x \to -\infty} [\tan^{-1} x] = -\frac{\pi}{2}$

FIND $f'(x)$ GIVEN $f(x) = \tan^{-1}(x)$

$y = \tan^{-1}(x)$

$x = \tan y$

$\frac{d}{dx}[x] = \frac{d}{dx}[\tan y]$

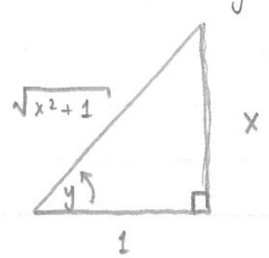
$1 = \sec^2 y \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \cos^2 y$

$\frac{dy}{dx} = \left[\frac{1}{\sqrt{x^2+1}}\right]^2$

$\frac{dy}{dx} = \boxed{\frac{1}{x^2+1}}$

$x = \tan(y)$



* $\boxed{\therefore \frac{d}{dx}[\tan^{-1} x] = \frac{1}{x^2+1}}$

AND

* $\boxed{\therefore \int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C}$

NOTE: $\frac{d}{dx}\left[\cot^{-1} x\right] = -\frac{1}{x^2+1}$

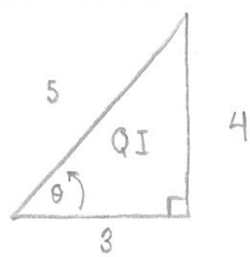
$$\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left[\csc^{-1} x\right] = -\frac{1}{|x|\sqrt{x^2-1}}$$

EX#1) SIMPLIFY.

a) $\csc(\underbrace{\tan^{-1}\tfrac{4}{3}}_{\theta})$

$\boxed{\csc\theta = \tfrac{5}{4}}$



$\theta = \tan^{-1}\frac{4}{3}$

θ IN QI

b) $\sec(\underbrace{2\tan^{-1}\tfrac{2}{3}}_{2\theta})$

$= \sec 2\theta$

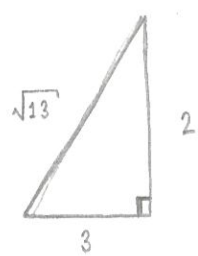
$= \dfrac{1}{\cos 2\theta}$

$= \dfrac{1}{\cos^2\theta - \sin^2\theta}$

$= \dfrac{1}{\left(\frac{3}{\sqrt{13}}\right)^2 - \left(\frac{2}{\sqrt{13}}\right)^2}$

$= \dfrac{1}{\frac{9-4}{13}}$

$= \dfrac{13}{5}$



$\theta = \tan^{-1}\frac{2}{3}$

θ IN QI

RECALL: $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

HWK #33: $\int \frac{e^x}{e^{2x}+1}\,dx$

$$\int \frac{e^x}{(e^x)^2+1}\,dx = \int \frac{1}{u^2+1}\,du = \tan^{-1} u + C = \boxed{\tan^{-1}(e^x)+C}$$

let $u = e^x$

$du = e^x\,dx$

NOTE: $\int \left[\frac{e^{2x}}{e^{2x}+1}\right] = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln(e^{2x}+1)+C}$

let $u = e^{2x}+1$

$du = e^{2x}\cdot 2\,dx$

$\frac{1}{2}du = e^{2x}\,dx$

* IF NUMERATOR & DENOMINATOR HAVE THE SAME e^x TERM, INTEGRAL IS $\ln x$, ELSE $\tan^{-1} x$

EX #2) $\int \frac{1}{x^2+a^2}\,dx$

$$= \int \left[\frac{1}{x^2+a^2}\right]\cdot \frac{\frac{1}{a^2}}{\frac{1}{a^2}}\,dx$$

$$= \int \frac{\frac{1}{a^2}}{\left(\frac{x}{a}\right)^2+1^2}\,dx$$

$$= \frac{1}{a^2}\int \frac{1}{\left(\frac{x}{a}\right)^2+1^2}\,dx$$

let $u = \frac{x}{a}$

$du = \frac{1}{a}\,dx$

$a\,du = dx$

$$= \frac{\cancel{a}}{a^{\cancel{2}}}\int \frac{1}{u^2+1}\,du = \frac{1}{a}\left[\tan^{-1} u\right] + C = \boxed{\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C}$$ *

EX#3) $\displaystyle\int \frac{1}{x^2+5}\,dx$

$$= \int \frac{1}{x^2+(\sqrt{5})^2}\,dx$$

$$= \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

$$= \boxed{\frac{\sqrt{5}}{5} \tan^{-1}\left(\frac{\sqrt{5}\,x}{5}\right) + C}$$

<u>HWK # 41:</u> $\displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{\tan^{-1} x}}{x^2+1}\,dx$

let $u = \tan^{-1} x$

$du = \dfrac{1}{x^2+1}\,dx$

$x = \sqrt{3} \Rightarrow u = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$x = 1 \Rightarrow u = \tan^{-1}(1) = \frac{\pi}{4}$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{u}\,du = \frac{2}{3} u^{\frac{3}{2}} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{2}{3}\left[\left(\frac{\pi}{3}\right)^{\frac{3}{2}} - \left(\frac{\pi}{4}\right)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[\frac{\pi\sqrt{\pi}}{3\sqrt{3}} - \frac{\pi\sqrt{\pi}}{8}\right]$$

$$= \boxed{\frac{2\pi\sqrt{\pi}}{3}\left[\frac{1}{3\sqrt{3}} - \frac{1}{8}\right]}$$

EX 4) $\displaystyle\int_{\ln 3\sqrt{3}}^{\ln 3} \left[\frac{e^x}{e^{2x}+9}\right] dx$

$u = e^x$ $\qquad$ $x = \ln 3 \to u = e^{\ln 3} = 3$

$du = e^x\,dx$ $\qquad$ $x = \ln 3\sqrt{3} \to u = e^{\ln 3\sqrt{3}} = 3\sqrt{3}$

$$\int_{3\sqrt{3}}^{3} \frac{1}{u^2+9}\,du = \frac{1}{3}\tan^{-1}\frac{u}{3}\Big|_{3\sqrt{3}}^{3}$$

$$= \frac{1}{3}\left[\tan^{-1}(1) - \tan^{-1}(\sqrt{3})\right]$$

$$= \frac{1}{3}\left[\frac{\pi}{4} - \frac{\pi}{3}\right] = \frac{1}{3}\left[\frac{3\pi - 4\pi}{12}\right] = \boxed{\frac{-\pi}{36}}$$

HWK #42: $\displaystyle\int_{1}^{\sqrt{e}} \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$

$u = \ln x$ $\qquad$ $x = \sqrt{e} \to u = \ln\sqrt{e} = \frac{1}{2}$

$du = \frac{1}{x}\,dx$ $\qquad$ $x = 1 \to u = \ln(1) = 0$

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u\Big|_0^{\frac{1}{2}}$$

$$= \sin^{-1}\left(\tfrac{1}{2}\right) - \sin^{-1}(0)$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

HWK #43: $\displaystyle\int_1^3 \frac{1}{\sqrt{x}(x+1)}\,dx = 2\int_1^{\sqrt{3}} = \frac{1}{u^2+1}\,du = 2\tan^{-1}u\Big|_1^{\sqrt{3}}$

$$= 2\left[\tan^{-1}\sqrt{3} - \tan^{-1}1\right]$$

$$= 2\left[\frac{\pi}{3} - \frac{\pi}{4}\right]$$

$$= 2\left[\frac{\pi}{12}\right]$$

$$= \frac{\pi}{6}$$

$u = \sqrt{x}$ $\qquad$ $x = 3 \to u = \sqrt{3}$

$du = \frac{1}{2\sqrt{x}}$ $\qquad$ $x = 1 \to u = \sqrt{1} = 1$

$2\,du = \frac{1}{\sqrt{x}}$

$x = u^2$