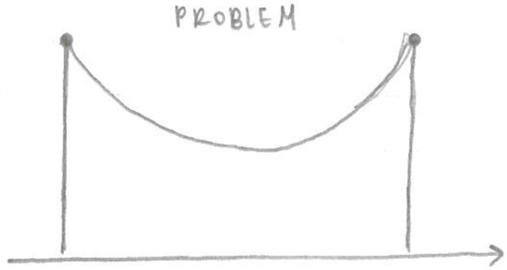
6.8 HYPERBOLIC FUNCTIONS AND HANGING CABLES

06/06

HANGING CABLE PROBLEM

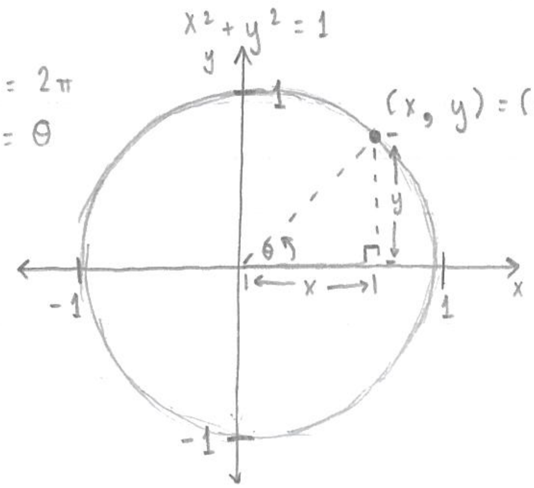


HYPERBOLIC FUNCTIONS come from the unit hyperbola. $\rightarrow x^2 - y^2 = 1$

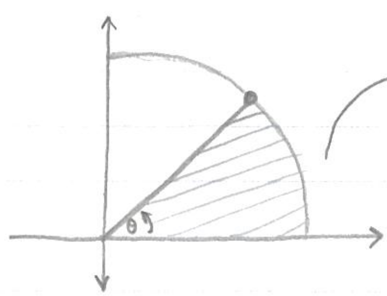
TRIGNOMETRIC FUNCTIONS come from the unit circle. $\rightarrow x^2 + y^2 = 1$

$C_\cup = 2\pi$

$S_\cup = \theta$



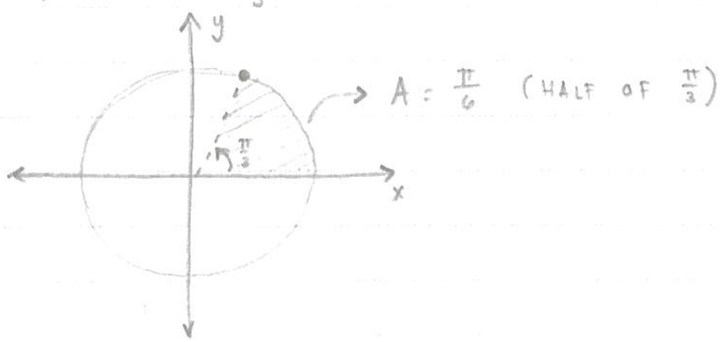
$S = r\theta$



$A = \frac{1}{2} r^2 \theta$

$A_u = \frac{\theta}{2}$

SUPPOSE $\theta = \frac{\pi}{3}$



IN THE UNIT CIRCLE,

$$A = \frac{\theta}{2}$$

- OR -

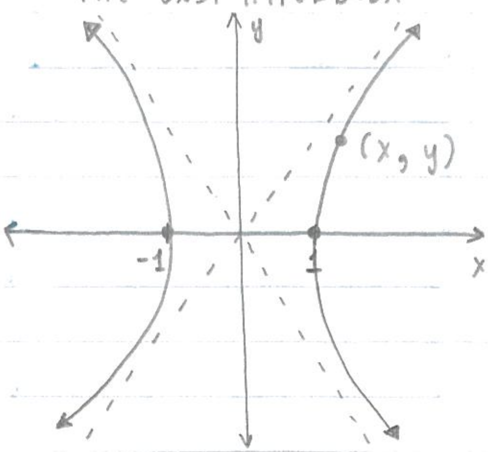
$$2A = \theta$$

IF WE DEFINED A PARAMETER "t" TO REPRESENT THE AREA OF A SECTOR IN THE UNIT CIRCLE, WE CAN SAY:

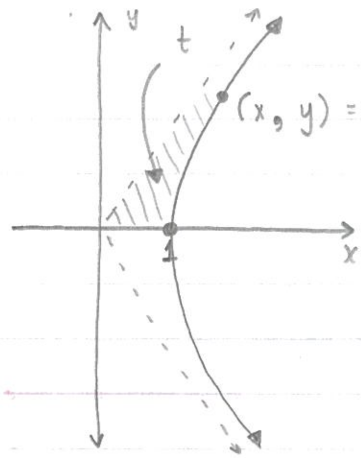
$\sin t$ WHEN $t = \frac{\pi}{6}$ IS THE SAME
AS $\sin\theta$ WHEN $\theta = \frac{\pi}{3}$

$$\sin t = \sin\theta$$
$$t = \frac{\pi}{6} \quad \theta = \frac{\pi}{3}$$

THE UNIT HYPERBOLA

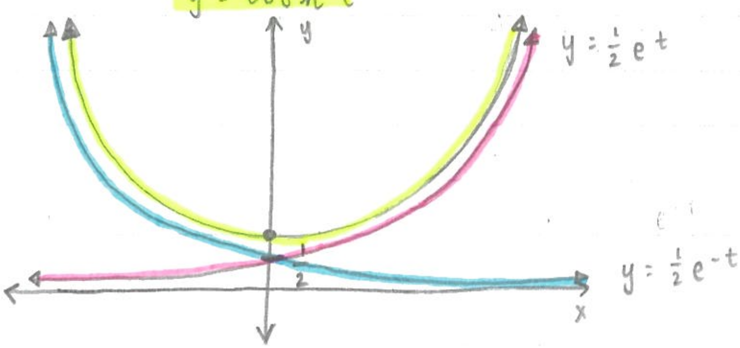


$\ast \ \boxed{x^2 - y^2 = 1}$



IF $t = 0$,
THEN $\cosh 0 = 1$
AND $\sinh 0 = 0$

$y = \cosh t$



t	cosh t
↑	$+\infty$
+	> 1
0	1
−	> 1
↓	$+\infty$

$\ast \ \boxed{\cosh t = \dfrac{e^t + e^{-t}}{2}}$

IF WE DIFFERENTIATE W.R.T. "t"

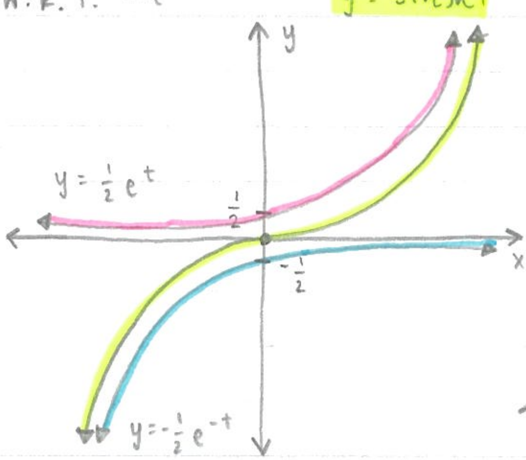
$\dfrac{d}{dt}\left[\cosh t\right]$

$\dfrac{d}{dt}\left[\dfrac{e^t + e^{-t}}{2}\right]$

$= \dfrac{e^t - e^{-t}}{2}$

$= \boxed{\sinh t}$

$y = \sinh t$



t	sinh t
↑	$+\infty$
+	> 1
0	0
−	< 1
↓	$-\infty$

$\ast \ \boxed{\sinh t = \dfrac{e^t - e^{-t}}{2}}$

$$\int \sinh t \, dt$$

$$\int \frac{e^t - e^{-t}}{2} \, dt$$

$$= \frac{1}{2} \int e^t - e^{-t} \, dt$$

$$= \frac{1}{2} \left[e^t - \left(\frac{e^{-t}}{-1} \right) \right] + C$$

$$= \frac{e^t + e^{-t}}{2} + C = \boxed{\cosh t + C}$$ ☆ ☆

NOTE:

☆ $\boxed{\frac{d}{dx}[\sinh x] = \cosh x}$

☆ $\boxed{\frac{d}{dx}[\cosh x] = \sinh y}$

☆ $\boxed{\frac{d}{dx}[\tan x] = \operatorname{sech}^2 x}$

EX#1) $$\int_0^{\ln\frac{1}{2}} \frac{4}{(e^x + e^{-x})^2} \, dx$$

$$= \int_0^{\ln\frac{1}{2}} \left[\frac{2}{e^x + e^{-x}} \right]^2 dx$$

$$= \int_0^{\ln\frac{1}{2}} \left[\frac{1}{\cosh x} \right]^2 dx$$

$$= \int_0^{\ln\frac{1}{2}} \operatorname{sech}^2 x \, dx$$

$$= \tanh x \Big|_0^{\ln\frac{1}{2}}$$

$$= \frac{e^x - e^{-x}}{e^x + e^{-x}} \Big|_0^{\ln\frac{1}{2}}$$

$$= \left[\frac{e^{\ln\frac{1}{2}} - e^{-\ln\frac{1}{2}}}{e^{\ln\frac{1}{2}} + e^{-\ln\frac{1}{2}}} - \cancel{\frac{e^0 - e^0}{e^0 + e^0}} \right]$$

$$= \left[\frac{\frac{1}{2} - 2}{\frac{1}{2} + 2} \right]$$

$$= \frac{-\frac{3}{2}}{\frac{5}{2}}$$

$$= -\frac{3}{5}$$

EX #2) SHOW THAT $\frac{d}{dx}[\tanh x] = \operatorname{sech}^2 x$.

$\frac{d}{dx}[\tanh x]$

$= \frac{d}{dx}\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]$

$= \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$

$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - (\cancel{e^{2x}} - 2 + \cancel{e^{-2x}})}{(e^x+e^{-x})^2}$

$= \frac{4}{(e^x+e^{-x})^2}$

$= \left[\frac{2}{e^x+e^{-x}}\right]^2$

$= \left[\frac{1}{\cosh x}\right]^2$

$= \boxed{\operatorname{sech}^2 x}$

☆ NOTE: $\boxed{\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x}$

$\boxed{\frac{d}{dx}[\operatorname{csch} x] = -\operatorname{csch} x \coth x}$

$\boxed{\frac{d}{dx}[\coth x] = -\operatorname{csch}^2 x}$

NOTE: $\cosh x + \sinh x$

$= \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$

$= \frac{2e^x}{2}$

$= e^x$

$\cosh x - \sinh x$

$= \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}$

$= \frac{-2e^{-x}}{2}$

$= -e^{-x}$

☆ $\cosh^2 x - \sinh^2 x = 1$

EVEN: $\cosh(-x) = \cosh x$

ODD: $\sinh(-x) = -\sinh x$

EX#3) SOLVE FOR X.

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = -\frac{1}{2}$$

$$2(e^x - e^{-x}) = -(e^x + e^{-x})$$

$$2e^x - 2e^{-x} = -e^x - e^{-x}$$

$$3e^x - e^{-x} = 0 \qquad \Rightarrow \text{LCD: } e^{-x}$$

$$3e^{2x} - 1 = 0$$

$$3e^{2x} = 1$$

$$e^{2x} = \frac{1}{3}$$

$$2x = \ln\frac{1}{3}$$

$$\boxed{x = \frac{1}{2}\ln\frac{1}{3}}$$

EX#4) SOLVE FOR X.

$$e^x\,[e^x - e^{-x}] = [2y]\,e^x$$

$$e^{2x} - 1 = 2ye^x$$

$$e^{2x} - 2ye^x - 1 = 0$$

let $u = e^x$

$$u^2 - 2yu - 1 = 0$$

$$u = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$$

$$u = \frac{2y \pm 2\sqrt{y^2 + 1}}{2}$$

$$u = y \pm \sqrt{y^2 + 1}$$

$$\Longrightarrow \quad e^x = y + \sqrt{y^2 + 1}$$

$$\boxed{x = \ln\left|y + \sqrt{y^2 + 1}\right|}$$

EX #5) $\int_0^{\ln\frac{1}{2}} \tanh^2 x \,\operatorname{sech}^2 x \, dx$

$u = \tanh x$

$du = \operatorname{sech}^2 x \, dx$

$x = \ln\frac{1}{2} \rightarrow u = \tanh\left(\ln\frac{1}{2}\right) = \frac{e^{\ln\frac{1}{2}} - e^{-\ln\frac{1}{2}}}{e^{\ln\frac{1}{2}} + e^{-\ln\frac{1}{2}}} = \frac{-\frac{3}{2}}{\frac{5}{3}} = -\frac{3}{5}$

$x = 0 \rightarrow u = \tanh(0) = 0$

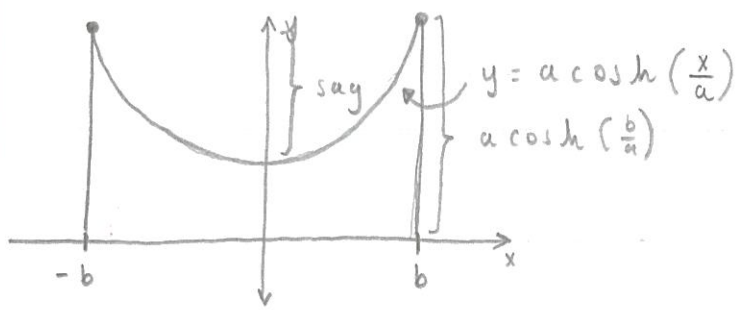
$$\int_0^{-\frac{3}{5}} u^2 \, du = \frac{1}{3}\left[u^3 \Big|_0^{-\frac{3}{5}} \right]$$

$$= \frac{1}{3}\left[\left(-\frac{3}{5}\right)^3 - (0)^3 \right]$$

$$= \frac{1}{3}\left[-\frac{27}{125} \right]$$

$$= \boxed{-\frac{9}{125}}$$

HWK #70:



$\mathcal{L} = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx$

$f(x) = a\cosh\left(\frac{x}{a}\right)$

$f'(x) = a\sin\left(\frac{x}{a}\right) \cdot \frac{1}{a}$

$f'(x) = \sin\left(\frac{x}{a}\right)$

$$L = 2\int_0^b \sqrt{1 + \left[\sinh\left(\frac{x}{a}\right)\right]^2} \, dx$$

$$= 2\int_0^b \sqrt{\cosh^2\left(\frac{x}{a}\right)} \, dx$$

$$= 2\int_0^b \cosh\left(\frac{x}{a}\right) dx$$

$$= 2\left[a\sinh\left(\frac{x}{a}\right) \Big|_0^b \right]$$

$$= 2a\left[\sinh\left(\frac{x}{a}\right) - \cancel{\sinh(0)}^{0}\right]$$

$$L = 2a\sinh\left(\frac{x}{a}\right)$$

b) $sag(s) = a\cosh\left(\frac{b}{a}\right) - a$